Conceptions of intuition in Poincaré's philosophy of mathematics

Olga Pombo
opombo@fc.ul.pt
Center for Philosophy of Science of the University of Lisbon,
Faculty of Sciences, University of Lisbon

We know that intuition is a very ambiguous word, a word which we use to designate a capacity or a faculty which we are not able to describe with sufficient accuracy.

From Plato's *noesis* to Aristotle's *nous*, from Descartes' intellectual intuition to Kant's sensible and pure intuition, from Spinoza's *scientia visionis* to Bergson's vital intuition, from Schelling's metaphysical intuition to Husserl's categorical intuition, the word intuition is used with very different meanings. And the etymological significance of the word as a direct, immediate vision does not help us solve the issue as we still would need to understand what is the nature of such a vision (a quasi-perception, an intellectual vision), what does such a vision gives us to see (a real individual object, an ideal entity, an essence, a concept, a relation) or what is the cognitive value of such a vision (a method for valid knowledge, a doubtful process, a discovery device, a guidance for action).

We also know that intuition plays a major role in Poincaré's Philosophy of Mathematics. So important, that Poincaré is usually considered a pre-intuitionist, someone who, before Brower, could consider mathematics as a free creation of the human mind.

However, Poincaré uses the concept of intuition in a great variety of meanings. He is aware of the plurivocity of the word. He is aware of the need for distinguishing different types of intuition. And he even gives some precise indications in this respect. That is the case of a celebrated distinction between three types of intuition he formulates in *La Valeur de la Science*. As he writes:

> "Nous avons donc plusieurs sortes d'intuitions; d'abord, l'appel aux sens et à l'imagination; ensuite, la généralisation par induction, calqué pour ainsi dire, sur les procédés des sciences

> expérimentales ; nous avons enfin l'intuition du nombre pur, celle d'où est sortie le second des axiomes que j'énonçais toute à l'heure et qui peut engendrer le véritable raisonnement mathématique» (VS : 33).

But, in the very same text, and also in other writings, Poincaré uses the word intuition with different meanings. And, even within what could be called one specific type of intuition - we will see this later - he introduces different formulations pointing to diverse conceptions of intuition. That is why it seems legitimate to state that Poincaré provides neither a clear doctrine nor a systematic use of the concept of intuition.

The aim of this paper is precisely to contribute to the identification and characterization of the various types of intuition put forward by Poincaré taking his texts as a laboratory for looking for what intuition might be. I will stress that these diverse conceptions were mainly formulated in the context of Poincaré's controversies in opposition to Logicism, to Formalism and in the context of Poincaré's very peculiar Conventionalism. I will try to demonstrate that, in each case, Poincaré comes close to a specific tradition (Kant, of course, but also Leibniz and Peirce).

1. Against Logicism

The main opposition concerns the fact that for Poincaré mathematics cannot be reduced to Logic and that precisely because intuition plays a significant role. In his text "Les Mathématiques et la Logique", Poincaré begins by pointing out his complete opposition to such a thesis.

Five arguments are put forward. Let us begin with the first three. **The first** if of factual nature. The logicist program is evaluated by its results, that is, by the contradictory consequences of its own development. As Poincaré declares:

> "Malheureusement, ils sont arrives à des résultats contradictoires, c'est ce qu'on appelle les antinomies cantoriennes" (SM: 127).[1]

[1] In fact, the argument is not very convincing since the example of Cantor could apply for Cantor and not necessarily to all Logicism. For a study on the polemics of Poincaré against the logicism, cf. Goldfarb (1985),

A similar **second** argument is developed in what concerns historical origins of mathematics. If mathematics could have been made just with Logic and by Logic, its development would have not taken place.[2] That is, Logic guarantees the rigor of mathematics but not its invention and progress.

The **third** is a pedagogical argument. For reasons parallel to those of historical development of mathematics, it is not possible to learn and to teach mathematics without the help of extra-logical elements. As Poincaré states repeatedly, logicist methodology for learning mathematics in which there is no place for any kind of intuition is "contrary to all reasonable psychology".[3] That is, Logic guarantees the rigor of mathematics but does not provide comprehension.

Several types of intuition are involved in this process of comprehension. To comprehend, students need to see with the eyes of the body. As Poincaré writes in *Les Définitions Mathématiques et l'Enseignement*:

> "Dans les écoles primaires, pour définir une fraction, on découpe une pomme " (SM : 106).

Clearly, Poincaré points here to **sensible intuition.** But that need of a sensitive, direct contact with the material entities corresponds to a very preliminary moment, both at the level of individual comprehension, and at the level a collective historical process of mathematical development".[4] Soon, this need of sensible images has to be surmounted. As in the history of mathematics, students have to realize, and even to desire, to rise above that initial, rudimentary, primitive level of comprehension.

> "Mais il faudrait leur montrer qu'ils ne comprennent pas ce qu'ils croient comprendre, les amener à se rendre comte de la grossièreté de leur concept primitif, à désirer d'eux-mêmes qu'on l'épure et le dégrossisse » (SM : 107).

We understand well the fundamental reason why, for Poincaré, sensible intuition cannot have any relevant role in mathematics, neither in its

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[2] As Poincaré writes : "Si les mathématiques n'en avaient pas d'autre [syllogism] elles seraient donc tout de suite arrêtées dans leur dévelopment » (SH: 38). In another passage, Poincaré writes : "Ce n'est certainement pas comme cela que l'esprit humain a procédé pour construire les mathématiques" (SM: 126).

[3] "Cette méthode est évidemment contraire à toute saine psychologie" (SM : 126).

[4] In fact, Poincaré believes in a non demonstrated assumption according to which there is an isomorphism between filogenesis and ontogenesis. As he writes, "les questions se poseront successivement à l'enfant, comme elles se sont posées successivement à nos pères » (SM : 112

learning and comprehension, nor in its historical development. And it should be so because, for Poincaré, mathematics is independent of material entities. To exist means to be without contradiction.

Now, to comprehend mathematics, Poincaré stresses further, students should fulfill their need of thinking with images:

> "Sous chaque mot, ils veulent mettre une *image sensible*" (SM: 105, our emphasis).

But, what does Poincaré mean by sensible image ("image sensible")? What kind of image is Poincaré pointing to? Note that for Poincaré it is not just young students who show such necessity of thinking with images. Poincaré extends such a need to all men. As he states:

> "Comment trouver un énoncé concis qui satisfasse à la fois aux règles intransigeants de la logique (…) et à *notre* besoin de penser avec des images » (SM : 114, our emphasis).

Of course, when Poincaré speaks of "our need of thinking with images" he is not talking about images provided by sensible intuition. That would correspond to a very rudimentary level of both individual and collective mathematical comprehension. Poincaré is surely pointing to another type of intuition corresponding to the production of other kinds of images on the basis of which the students (and all of us) think. Let us quote again:

> "Sous chaque mot, ils veulent mettre une *image sensible.* (…) A cette condition seulement ils comprendront et ils retiendront. Ceux là souvent se font illusion à eux-mêmes ; ils n'écoutent pas les raisonnements, ils regardent les *figures* " (SM: 105, our emphasis).

I suggest that it is necessary to make here a distinction of, at least, three levels. In fact, it is not the same a) to see an apple being cut (sensible intuition), b) to see the correspondent drawing of an apple cut in several slices (figure), c) to represent it imagetically, without the presence of the apple or the drawing done with pencil on paper (**imaginative intuition**).

The (a) sensible intuition supposes the presence of a singular, individual entity (be it an apple) or the observation of a determined particular event (the cutting of the apple). The (b) drawing (with pencil on paper) of a figurative representation (a curve, a ring, a circle, a triangle, etc) corresponds to the second level of that sensible intuition because now what is seen is not the individual in its particularity, but the iconic representation of that individual sketched on paper. There is a huge difference between the individual entity (the apple) and its already more or less symbolic, more or less schematic, more or less abstract representation (the drawing). However, it is yet with the eyes of the body that one sees that figure sketched in the exterior world (the paper). So, we may say that we are still at the level of sensible intuition, even if of diagrammatic nature.

Now, (c) imaginative intuition is the spontaneous capacity for producing an imagetic representation of a circular figure regardless of the presence of any circular object (any apple) and of any drawing (any curve, ring, circular figure). This is a much more complex process since, when we see an individual object (sensible intuition) or a specific drawing empirically produced in the external world (sensible intuition of diagrammatic nature), we are always facing to concrete entities, endowed with particular features and dimensions (the drawing of a circle is always a determined entity with determined dimensions, the drawing of a point is always a very small entity with very small dimensions). Now, with imaginative intuition the image is produced by our imagination independently of any empirical entity, just in accordance with the concept of the referred entity. That is what happens in Kant: imagination spontaneously produces the image which will be further synthesized with the concept. The construction of mathematical entities in Kant requires only the pure intuition of time within which imagination produces the image of that entity, that is, that production is made fully *a priori*, without any support coming from experience.

The interesting question now, would be: is it possible to make geometry by keeping away from any image, that is, not only without any sensible image (be it an apple or a drawing), but also without any image at all, that is, without any image produced by our imaginative intuition?

For Kant, we cannot. And we cannot because Geometry needs images produced by imagination. However, for Kant, what is necessary is neither the sensible image of an apple, nor the empirical, diagrammatic drawing of any round ring, but the image produced by imagination within the pure *a priori* intuition of space and time, more precisely the rule for constructing those images (the *schemata*).

For Brower too, we cannot. And we cannot for a much more radical reason: because intuition is the only basis, the very source for the construction of mathematics.

On the contrary, for Bolzano, yes we can make geometry without any images at all. We can and in fact we have to do it because our imagination is unable to produce images corresponding to pure concepts such as mathematic idealities.[5]

Poincaré does not formulate this crucial question, nor does he faces it in its radicalism. We cannot know exactly what his answer would be. At one point, he seems to approach this question when he writes:

> "Un grand avantage de la géométrie, c'est précisément que les sens y peuvent venir au secours de l'intelligence (…). Malheureusement, nos sens ne peuvent nous mener bien loin. Et ils nous faussent compagnie dès que nous voulons nous envoler en dehors des trois dimensions classiques » (SM : 38-39).

But Poincaré is very ambiguous. On the one hand, he recognizes, against Hilbert, that the sensible intuition may be of the same value in Geometry; but, on the other hand, he stresses that its role is limited, merely introductory, and only in the context of Euclidean geometry. Even so, let me quote a passage which may give us some insight into his position:

> "Nous sommes dans la classe de 4ème ; le professeur dicte : le cercle est le lieu des points du plan qui sont á la même distance d'un point intérieur appelé centre. Le bon élève écrit cette phrase sur son cahier ; le mauvais élève y dessine des bonhommes ; mais ni l'un ni l'autre n'ont compris; alors le professeur prend la craie et trace un cercle sur le tableau. « Ah ! pensent les élèves, que ne disait-il tout de suite : un cercle c'est un rond. Nous aurions compris ! » (SM : 107).

And Poincaré adds a last brilliant sentence:

> « Sans doute, c'est le professeur qui a raison. (…). Mais il faudrait leur montrer qu'ils ne comprennent pas ce qu'ils croient comprendre » (ibid).

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[5] In fact, for Bolzano imagination does not have the conditions for producing any image corresponding to geometrical concepts and theorems. As he writes : "les lignes que notre imagination parvient à nous représenter, ne sont pas infiniment longues » (Bolzano, 2010: 137).

The entire quotation is very interesting. But it is to the last sentence that I would like to draw your attention to. What do the students suppose they understand? What do they really understand? What should the teacher make them be aware of?

What I would like to stress is that the answer of Poincaré to the question formulated above - is it possible to make geometry by keeping away from any image, not only any sensible image (an apple or a drawing), but also any image at all - may probably be detected in that last bright sentence.

Students presume to understand the circle in the sensible (diagrammatic) representation they see in the blackboard. But what they do really understand (when they see such figure) is the invisible, ideal circle whose expression may be seen as the inadequate shadow appearing on the blackboard. Of course, the drawing done by the teacher on the blackboard is not a circle. But the drawing done by the teacher on the blackboard is a sensible image of a diagrammatic nature, able to "express" the concept of a circle. That is why the students are able "to see the circle" when looking at the drawing done in the blackboard. The students may say (or they may even cry out) that they understand the circle **in** the drawing, but what they really understand is the circle itself (the concept of the circle) expressed **by** the drawing.

Three important consequences may be drawn from the passage given above.
1. Against Kant, and in accordance with Leibinz, Poincaré safeguards the objectivity of mathematics. 2. Poincare points to another kind of intuition which I propose to designate as **conceptual intuition**, precisely the knowledge of objective mathematical entities. Again, it is Leibniz and not Kant who can be recognized behind this capacity. 3. Poincaré recognizes the expressive capacity of images. Once more, it is Leibniz and not Kant who understood deeply what an image is and the secrecy of its expressive nature[6].

Unfortunately, Leibniz's awareness of what an image is, turns to become impossible after Kant established the rupture between sensibility and understanding. When recognizing the cognitive role of image, Poincaré might suppose that he is retrieving Kant, but it is Leibniz's acute

[6] For a study on the constitutive role of symbolism in Leibniz, cf. Pombo (2010).

understanding of what an image is, that is at work here. This is what I propose to call Poincaré's conceptual intuition.

Finally, another type of intuition is involved in the process of comprehension of mathematics. It responds to the fact that students (as well as all other men) need to understand the order of the deductive chain

> "Ils veulent savoir, non seulement si tous les syllogismes d'une démonstration sont corrects, mais pourquoi ils s'enchaînent dans tel *ordre*, plutôt que dans tel autre"(SM: 105, our emphases).

It is easy to signalize here another conception of intuition, interior so to say to the demonstration itself, a sudden capacity of seeing at a glance, *"d'un seul coup d'oeil" (SH: 40),* the order of the demonstrative chain.

> "Une démonstration mathématique n'est pas une simple juxtaposition de syllogismes, ce sont les syllogismes placés dans une certaine *ordre* et l'ordre dans lequel ces éléments sont placés est beaucoup plus important que ne le sont ces éléments eux-mêmes. Si j'ai le *sentiment, l'intuition pour ainsi dire, de cet ordre*, de façon à apercevoir *d'un coup d'œil l'ensemble* du raisonnement, je ne dois plus craindre d'oublier l'un des éléments ; chacun d'eux viendra se placer lui-même dans le cadre qui lui est préparé" (SM : 45-46, our emphases).

We are here facing the well known and much commented **demonstrative or architectonical intuition** (we could say), which enables us to grasp the order of the inferences, the general scheme of the demonstrative movement, or as Poincaré says, *"la marche générale du raisonnement" (SM: 45)*.

Poincaré refers to this type of intuition as a sentiment, a delicate feeling, hard to define, *"un sentiment délicat, et difficile à definir" (SM: 46).* But what is important to mention is that this demonstrative, architectonical intuition, this intuition of mathematical order *"intuition de l'ordre mathématique" (SM: 46)* – coincides, at its highest level, with inventive activity. As Poincaré explains, it is because some people are lacking this sentiment, that they are unable to comprehend some more elevated mathematics. It is because others have this kind of intuition with some intensity, that they are able to comprehend mathematics. And it is because some few others have this capacity in high level, that they are able, not only to comprehend, but to create new mathematics:

« Les uns ne posséderons *ce sentiment délicat, et difficile à définir* (…) et alors ils seront incapables de comprendre les mathématiques un peu élevées (…). D'autres n'auront ce sentiment qu'à une faible degré (…) ils pourrons comprendre les mathématiques et quelque fois les appliquer (…). Les autres enfin posséderont a un *plus ou moins haute degré l'intuition spéciale* dont je viens de parler et alors, nom seulement ils pourront comprendre les mathématiques (…) mais ils pourront devenir créateurs et chercher à inventer avec plus ou moins de succès, suivant que cette intuition est chez eux plus ou moins développée. (SM : 46, our emphases).

Therefore, this demonstrative or architectonical intuition, at its highest level, becomes a special intuition, an "intuition special", that is, an **inventive intuition,** a kind of intuition we will meet again, further on.

Let us now come to the **4th argument** put forward by Poincaré against Logicism. We are now facing a psychological argument, mostly formulated against Russell[7]. It concerns the principle of induction which, for Poincaré, is the marvelous result of the activity of human spirit and not, as Russell defended, just the definition of entire number.

Against Russell, who sustained that the richness of mathematics is independent from the (obscure, fragile) power of human spirit, against Russell for whom mathematical induction is not truly an induction because it does not go from the particular to the general since it is always (and already) within the general, Poincaré stresses the powerful virtue of mathematical induction, which, contrary to what happens in other sciences, is able to go from the particular to the general without losing necessity:

« L'induction, appliquée aux sciences physiques, est toujours incertaine, parce qu'elle repose sur la croyance à un ordre général de l'Univers, ordre qui est en dehors de nous. L'induction mathématique, c'est-à-dire la démonstration par récurrence, s'impose au contraire nécessairement, parce qu'elle n'est que l'affirmation d'une propriété de l'esprit lui-même » (SH : 42).

[7] For a study on this controversy, see Heinzmann (1994).

Poincaré also stresses that, unlike the rigorous but sterile syllogistic deduction[8], mathematical induction allows reaching a conclusion more general than the premises:

> « La vérification diffère précisément de la véritable démonstration, parce qu'elle est purement analytique et parce qu'elle est stérile. Elle est stérile parce que la conclusion n'est que la traduction des prémisses dans un autre langage. La démonstration véritable est féconde au contraire parce que la conclusion y est en un sens plus générale que les prémisses » (SH : 34).

Mathematical induction is therefore much more than a series of successive additions as Russell argued. It is the possibility of indefinitely repeating that operation, that is to say, an operative device, a powerful tool of the human spirit endowed with a creative virtue (“*virtue créatrice*”, SH:32), or – as Poincaré explicitly says – an instrument allowing to pass from the finite to the infinite:

> “[Le raisonnement par récurrence est] un instrument qui permet de passer du fini à l’infini » (SH : 40).

It is is precisely at this point that this (psychological) argument becomes a constructivist **5th argument**. In fact, for Poincaré, mathematical induction is irreducible, both to the principle of contradiction[9] (on the basis of which we can go on with the analytical development of further and further syllogisms although never creating really new developments), and to the experience, on the basis of which we can go on approaching higher and higher levels of generality but never reach the infinity.

> « ce que l'expérience pourrait nous apprendre, c'est que la règle est vraie pour les dix, pour les cent premiers nombres par exemple, elle ne peut atteindre la suite indéfinie des nombres, mais seulement une portion plus ou moins longue mais *toujours limitée* de cette suite » (SH : 41, our emphases).

But, it is precisely there that, when we recognize the limits of logic (against Coutorat’s inflexible Logicism) and of experience (against Russell’s mathematical Empirism), that, according to Poincaré, we are obliged to recognize that splendid human capacity of enclosing the infinity in one only, single formula. As Poincaré writes:

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[8] “Le raisonnement syllogistique reste incapable de rien ajouter aux donnes qu’on lui fournit” (SH:31-32).

[9] «(…) la règle du raisonnement par récurrence est irréductible au principe de contradiction » (SH : 41).

"Le caractère essentiel du raisonnement par récurrence c'est qu'il contient, condensés pour ainsi dire en une formule unique, une infinité de syllogismes » (SH : 38/39).

«Pour y arriver, il faudrait une infinité de syllogismes, il faudrait franchir un abîme que la patiente de l'analyste, réduit aux seules ressources de la logique formelle, ne parviendra á combler » (SH : 40).

"il s'agit d'enfermer une infinité dans une seule formule" (SH: 41).

That is why mathematical induction is considered by Poincaré as an *a priori* synthetic judgment[10], able consequently to extend our knowledge, an outcome of a very specific type of intuition: a **constructive (Kantian) intuition**.

That is why, for Poincaré, Mathematics is a constructive activity that goes beyond the limits of experience and of analysis. In fact, even if we do not have the capacity of seeing the infinity (with the eyes of the body – by sensible intuition) nor the capacity of imagining it, that is, of spontaneously producing an image of it (by imaginative intuition), however, we have the *a priori* conditions of possibility necessary for its possible iterative construction. As Poincaré writes in a crucial passage:

« Pourquoi donc ce jugement s'impose-t-il à nous avec une irrésistible évidence ? C'est qu'il n'est que l'affirmation de la puissance de l'esprit qui se sait capable de concevoir la répétition indéfinie d'un même acte *dès que cet acte est une fois possible*. L'esprit a de cette puissance une *intuition directe* et l'expérience ne peut être pour lui qu'une occasion de s'en servir et par là d'en prendre conscience » (SH : 41, our emphases).

There is here a kind of a processof a *mis en abîme*: the possibility of an iterative construction of infinity is rooted in the intuition of our *a priori* conditions of possibility. The intuition of those *a priori* conditions of possibility offers us the possibility of grasping our power of indefinitely iterating. As if the intuition of our own transcendental structure could

[10] « Cette règle, inaccessible à la démonstration analytique et à l'expérience, est le véritable type du jugement synthétique a priori » (SH : 41). This is a major point of controversy with Couturat (1913) for whom there is no place for synthetic judgments in mathematics (cf. SM: 127). For further criticism on Couturat, see also SM : 153-169.

open for us (or better, would be the occasion for) the possibility of seeing that infinite possibility. That is to say, against the logicism of Russell and Couturat, Poincaré goes back to Kant.

Poincaré is here in fact very close to Kant[11] and for two different reasons. First because he stresses, not only the capacity of human spirit of being responsible for the construction of mathematics, but the operative nature of that constructability, precisely by mathematical induction (or constructive intuition). As Kant would have said, it is true that we do not have an image of the infinite (our imagination is unable to produce it). But we have the rule for its production (the *schemata*). Similarly, Poincaré's conception of mathematical induction is endowed with an operative, constructive, conception of intuition which we proposed to name constructive intuition. Second, because additionally, Poincaré is aware of another kind of intuition - what he calls a **direct intuition** – through which the spirit becomes conscious of its own transcendental capacity for mathematical induction. Let us quote again the last sentence from the passage above:

> « L'esprit a de cette puissance une *intuition directe* et l'expérience ne peut être pour lui qu'une occasion de s'en servir et par là d'en prendre conscience [de soi même]» (SH : 41, our emphases).

As Kant would have said, we are here facing the apperception of the pure subject, that is, of the highest and ultimate transcendental structure and of its constitutive role.

Poincaré does not provide a full development of this highest level of direct intuition (or pure intuition, in Kantian terms). But, nevertheless the awareness of that highest level emerges here and there in his texts. That is what happens, in our view, when Poincaré emphasizes the unity which is necessary for mathematical construction:

> « Pour qu'une construction puisse être utile, pour qu'elle ne soit pas une vaine fatigue pour l'esprit, pour qu'elle puisse servir de marchepied à qui veut s'élever plus haut, il faut d'abord qu'elle possède une sorte *d'unité*, qui permette d'y voir autre chose que la juxtaposition de ses éléments » (SH : 44, our emphases).

[11] For a close study on the Kantian roots of Poincare's Philosophy of Mathematics, cf. Folina (1992).

As Kant would have said, if the spirit is able to grasp a unique formula for the multiplicity of elements, this is because it uses its own unity for reaching a higher level of construction; because, somehow, our spirit projects its own unity in what it constructs; because its own unity has a constitutive role.

The presence of Kant in Poincaré does not go that far. The awareness of the transcendental root of mathematical unity is a Kantian thesis which, as far as I know, Poincaré never defended. However, it is in line with Poincaré's argument in favour of mathematical unity.

2. Against formalism

The opponent is now Hilbert, even though, Poincaré highly praises the rigorous character of Hilbert's Geometry. [12] This does not prevent Poincaré from putting forward two main arguments agains it, each one related to a specific type of intuition.

The **first argument** is developed in a very straight controversial style. Poincaré criticizes Hilbert's program for several reasons: Hilbert does not need to know the meaning of the words and expressions he uses in mathematics; Hilbert does not need to know what the things referred by those words and expressions are; Hilbert only needs a system of linguistic equivalencies; for Hilbert a blind men could be a good geometer, in a word, for Hilbert mathematics might be done by machines. Poincaré does not hesitate to draw a kind of a caricature of Hilbert's program:

> « Pensons, dit Hilbert, trois sortes de choses que nous appellerons points, droites et plans, *convenons* qu'une droite sera déterminée par deux points et que, au lieu de dire que cette droite est déterminée par deux points, nous pourrons dire qu'elle passe par deux points ou que ces deux points sont situés sur cette droite. *Que sont ces choses*, non seulement nous n'en savons rien, mais nous ne devons pas

[12] For instance, about Hilbert's *Grundlagen der Geometrie*, Poincaré says : "un livre justement admiré et bien des fois couronné (…). Voila un livre don't je pense beaucoup de bien, mais que je ne recommanderais pas à un lycéen » (SM : 107).

> chercher savoir. Nous n'en avons pas besoin, et *quelqu'un qui n'aurait jamais vu* ni point, ni droite, ni plan, pourrait faire la géométrie tout aussi bien que nous » (SM : 128, our emphases)
>
> « Ainsi, bien entendue, pour démontrer un théorème, il n'est pas nécessaire ni même utile de savoir ce qu'il veut dire. On pourrait remplacer le géomètre par le *piano à raisonner* imaginé par Stanley Jevons (…) Pas plus que les *machines*, le mathématicien n'a besoin de *comprendre* ce qu'il fait » (SM : 128, our emphases)

Poincaré's claim is that mathematics cannot be reduced to the manipulation of a sign system. Mathematics is not a merely formal, empty language with its own syntax without any semantic reference to the world.[13]

Now, in this critical claim, intuition is thought by Poincaré as the root for the referential character of mathematics, that is, intuition is thought by Poincaré as a **meaning device**, as the origin for the meaning of words and expressions and for the understanding of what things are (or better, of what their relations are). We are facing here another important issue – the fact that, for Poincaré, the reference of mathematics to the world is not apparent as in Hilbert, but real. Even if, for Poincaré, the real means relational (again, the shadow of Leibniz is hovering around).

But now, we have to ask: what might this root be? What valuable advantage may intuition provide for establishing the connection of language to the things of the world, or better, to their relations? What might be the role of intuition when faced with to a formal system? Why is it impossible to replace the mathematician by the *piano à raisonner* of Stanley Jevons?[14]

In my view, the answer cannot be other than to say that intuition may provide the images needed to connect words and expressions of mathematics to the things and relations they signify. We need images because they allow us to see the things and the relations signified by the

[13] As Poincaré explains : "Croit-on que les mathématiques aient atteint la rigueur absolue sans faire de sacrifice ? pas du tout, ce qu'elles ont gagné en rigueur, elles ont perdue en objectivité. C'est en s'éloignant de la réalité qu'elles ont acquis cette pureté parfaite » (SM : 109).

[14] Poincaré refers to the Logic Piano constructed by the British economist and logician William Stanley Jevos (1835 –1882) and exhibited before the Royal Society in 1870.

words and expressions of mathematics. With Leibniz again, images are cognitive devices.

The **second argument** entails another distinction in Poincare's conception of intuition. As seen above, Poincaré praises Hilbert for the rigorous character of his Geometry, namely, he pays tribute to Hilbert's aim of reducing the number of axioms and of making the complete enumeration of the fundamental axioms of geometry. [15]But Poincaré does not believe in the possible accomplishment of this program. He considers it impossible to avoid some infiltration of non-explicit axiomatic suppositions or postulates into mathematical reasoning. And this is impossible because our esprit is active, alive, that is, because intuition has a deep and commonly hidden role in mathematics, precisely at the level of its fundamental axioms.

> "Il [Hilbert] voulait réduire au minimum le nombre des axiomes fondamentaux de la géométrie et en faire l'énumération complète ; or, dans les *raisonnements vivants*, pour ainsi dire, il est difficile de ne pas introduire un axiome ou un postulat qui passe *inaperçu*» (SM : 129, our emphases).

We are here facing another type of intuition. Let us summarize the situation. Poincaré is not anymore arguing against logicism in its more developed level (mathematics cannot be taught or learned or constructed on the basis logic alone, mathematics is a comprehensive task needing sensible intuition in its first steps (apples, drawings) and also spontaneous images produced by a Kantian imaginative intuition. Poincaré is not any more referring to conceptual intuition as objective knowledge of mathematical idealities, nor is he referring to the demonstrative, architechtonical intuition of the order, nor to mathematical induction as a kantian (constructive) intuition.

Poincaré is now arguing against formalism. But, he is not (as in his first argument) blaming formalism for its lack of intuition as a meaning device. Intuition is not anymore that which avoids the emptiness of words and mathematical expressions. Now, in this second argument against formalism, Poincaré requires a different concept of intuition. Now, he blames the formalist program for its inability to fully explain its

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[15] « Ce caractère formel de sa géométrie, je n'en fais pas de reproche à Hilbert. C'était là qu'il devait tendre, étant donné le problème qu'il se posait. Il voulait réduire au minimum le nombre des axiomes de la géométrie et en faire l'énumération complète » (SM : 128-129).

fundamental axioms. Now, intuition is thought as a procedure which necessarily infiltrates the foundations of mathematics.

That is, in the first argument against formalism, intuition is a **meaning device,** that which gives meaning to words and expressions of mathematics, a root for the understanding of what "things" are, the origin of mental images linking the words and expressions of mathematics to the things and relations they signify, an anti-mechanical remedy which negatively reacts to any attempt to reduce mathematics to a rigorous mechanical procedure.

In the second argument against formalism, intuition becomes a **foundational device**, that which infiltrates the foundations of mathematics, an active, energetic procedure, a living force of our mind, able to challenge the full explicitation efforts of a complete formalization.

Of course, it is still necessary to understand what it could be, how it works, this intuition as a foundational, full of life device. Poincaré does not answer this question, even if he frequently emphasizes the vital character of mathematics to which we can only gain access through intuition.

3. Against conventionalism

There is yet another argument **in the context of Poincaré's peculiar conventionalism.** And this argument (I would like to stress) embraces another important distinction in Poincare's conception of intuition.

Also formulated against formalism, the argument is of a hypothetical form. Poincaré invites us to *pretend* to accept the conventionalist solution according to which the fixation of the main mathematical suppositions (axioms) is done by convention. I quote again Poincaré's critique of Hilbert's program:

> « Pensons, dit Hilbert, trois sortes de choses que nous appellerons points, droites et plans, *convenons* qu'une droite sera déterminée par deux points et que, au lieu de dire que cette droite est déterminée par deux points, *nous pourrons dire* qu'elle passe par deux points ou que ces deux points sont situés sur cette droite. Que sont ces choses, non seulement nous n'en savons

> rien, mais nous ne devons pas chercher savoir. Nous n'en avons pas besoin, et quelqu'un qui n'aurait jamais vu ni point, ni droite, ni plan, pourrait faire la géométrie tout aussi bien que nous » (SM : 128, our emphasis).

But - he stresses - even if the conventionalist solution were fully possible, it would be legitimate to go on questioning the origin of such conventions.

> « *Admettons* même que l'on ait établie que toutes les théorèmes peuvent se déduire par des procédés purement analytiques, et que ces axiomes ne sont que des conventions. Le philosophe conserverait le droit de rechercher *les origines de ces convenions*, de voir pourquoi elles ont été jugées préférables aux conventions contraires » (SM : 129, our emphases).

Against conventionalists who precisely do not want to answer that question, or better, answer it with arbitrariness, Poincaré wants to find the origin of those conventions. And that origin takes us to an "instinct" which is guiding our choices. As Poincaré states:

> "Parmi toutes les constructions que l'on peut combiner avec les matériaux fournis par la logique, il faut faire un *choix*; le vraie géomètre fait ce choix judicieusement parce qu'il est guidé para un *sûr instinct*, ou par quelque *vague conscience* de je ne sais pas quelle *géométrie plus profonde, et plus cachée*, qui seule fait le prix de l'édifice construit» (SM : 129, our emphases).

Intuition is now a "sure instinct" which allows the mathematician to establish the foundations of mathematics not by pure, arbitrary convention, but by choosing those foundations. And the choice is made by an instinct - a guessing as Peirce would say. Intuition is here thought out as an anti-conventionalist device, an instinct enabling the mathematician to choose axioms instead of accepting pure arbitrary convention.

In addition, the choice is made by the mathematician on the basis of a "vague awareness" of a more "deep geometry" capable of giving value to the whole building. How should we understand this deeper level of mathematics in which is rooted the instinct, which allows the mathematician to choose the foundation of mathematics?

Are we here facing an almost **Platonic conception of intuition** as the possibility of accessing the deep, true, eternal roots of mathematics, which would constitute the origin of its foundations? This solution would be in line with Poincaré's rejection of Stuart Mill's empiricism, that is to say, in accordance with Poincaré's defense of mathematical objective (Platonic) idealities. Or, are we facing a kind of an **abductive intuition** giving access to a deep and hidden world within which the choice may be reasonably made? Are we facing a possible world in the context of which the choice gains reasonableness, an abductive universe in which, among many others, one specific choice appears as the more reasonable? Peirce would be now the great inspiration.

I suppose that this Peircean solution is more faithful to Poincaré's thought and more interesting for its operative value. Further, it may better illuminate that foundational, full of life role of intuition which we have mentioned above. In support of this interpretation, it is quite understandable that, precisely in the sequence of the above quotation, Poincaré comes to speak about the instinct as an instrument of invention:

> "Chercher l'origine de cet instinct, étudier les lois de cette géométrie profonde qui se sentent et que ne s'énoncent pas, ce serait encore une belle tâche pour les philosophes qui ne veulent pas que la logique soit tout. Mais ce n'est pas à ce point de vue que je veux me placer, ce n'est pas ainsi que je veux poser la question. [Ce que je veux dire c'est que] *Cet instinct dont nous venons de parler est nécessaire à l'inventeur* » (SM : 130, our emphases).

We know that Poincaré explicitly refers to **inventive intuition** in several texts. For instance, in *La Valeur de la Science*, he states:

> "La logique qui peut seule donner la certitude est l'instrument de la démonstration : l'intuition est l'instrument de l'invention » (VS : 37).

Now, that inventive intuition is thought as the "art of choosing among all the possible combinations"[16] and according to several main criteria, experience, pragmatic reasons, architectural and aesthetic principles like simplicity, elegance, symmetry, unity.[17]

[16] "L'art de choisir entre toutes les combinaisons possibles" (SM: 113)

[17] Cf., for instance, *L'Avenir des Mathématiques*, where Poincaré mentions "harmony, symmetry, balancing, order and unity" (SM: 29) and clearly connects these aesthetics criteria with the principle of economy. As he writes: "Cette satisfaction esthétique est par suite liée à l'économie de pensée » (SM : 30).

But this involves – as Peirce would have said – the capacity to choose the best combination.

> “Inventer, cela consiste précisément à ne pas construire les combinaisons inutiles et á construire celles qui sont utiles et qui ne sont qu’une infime minorité» (SM : 47).

We are here facing a major issue of Poincare’s Philosophy of Mathematics, namely the role of intuition and the relation between Logic and intuition. In the following passage, Poincaré makes this clear :

> « La logique nous apprend que sur tel out el chemin nous sommes sûr de ne pas rencontrer d’obstacle. Elle ne nous dit pas quel est celui qui mène au but. Pour cela**,** il faut *voir le but de loin* et *la faculté qui nous apprend á voir, c’est l’intuition*. Sans elle le géomètre serait comme un écrivant qui serait ferré sur la *grammaire*, mais qui n’aurait pas des *idées* (SM : 113, our emphases).

This passage is almost the same as in *La Valeur de la Science* where Poincaré writes :

> « L’analyse pure met à notre disposition une foule de procédés dont elle garantie l’infaillibilité (…). Mais, de tous ces chemins, quel este celui qui nous mènera le plus promptement au but ? Qui nous dira lequel il faut choisir ? Il nous faut une faculté qui nous fasse *voir le but de loin* et cette faculté, c’est l’intuition » (VS : 36).

And further on, on the same page, while comparing mathematics with chess game, Poincaré says that what is necessary for understanding the game,

> “C’est apercevoir la raison intime qui fait de cette série de coups successifs une sorte de *tout organisé* »(ibid, our empases).

This is a decisive moment. For inventing, we have to choose the best combination (on the basis of Peircean instinctive intuition). But, in order to make that choice we have to be able to see the whole – “*voir le but de loin”*. And, the faculty that teaches us to see is intuition – “*la faculté qui nous apprend á voir, c’est l’intuition”.*

For inventing, we need our ability to choose the best combination (**instinctive Peircean intuition**). But for inventing, we also need our capacity for seeing the organic whole ("*le tout organizé*"), that is, we need intuition **as aesthetic visibility**. Without such large visibility of the whole game we would have the rules (the grammar) but not the openness to the real world (the ideas).

> C'est par elle [l'intuition] que le monde mathématique reste en contact avec le monde réel et quand les mathématiques pures pourraient s'en passer, il faudrait toujours y avoir recours pour combler l'abîme qui sépare le symbole de la réalité » (SM : 112).

Poincaré is aware that such intuition (which we have named as aesthetic visibility) can only operate on the basis of a language opened to the world. That is why Poincaré cannot accept Peano's pasigraphy.[18] And that is why Poincaré would have accepted the Leibnizian project of a *Characteristica Universalis*. Indeed, in a very different temporal horizon and much beyond Peano's pasigraphy, Leibniz has looked for the possibility of constructing a philosophical language simultaneously rigorous and meaningful, a universal artificial language enriched by the meaning strategies operating in natural languages.[19] In other words, for Poincaré – as for Leibniz – the aim is to reconcile syntax and semantics, rigour and meaning.

This is obviously a gigantic and impossible project. But yet a desirable one. Let me quote Poincaré one last time, now from *L'Avenir des Mathématiques* :

> « En Mathématiques la rigueur n'est pas tout, mais sans elle il n'y a rien : une démonstration qui n'est pas rigoureuse c'est le néant. Je crois que personne ne contestera cette vérité. Mais si on la prends trop à la lettre, on serait amené a conclure qu'avant 1820, par exemple, il n'y avait pas de mathématiques » (SM : 31).

And, emphasizing again the role of aesthetic intuition, he adds:

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[18] Cf., for example, chap. VII de *Les Mathématiques et la Logique*, SM: 135-138.

[19] For that, Leibniz follows two parallel strategies: 1) logical – to construct a system of signs and operative rules allowing the rigorous expression of thought and its articulations; 2) semantical – to understand the constitutive mechanisms of natural languages which allow their openness to the world. For a developed presentation of the Leibnizian project, see Pombo (1987).

> « Ce serait manifestement excessif. Les géomètres de ce temps sous-entendait volontiers ce que nos expliquons par des prolixes discours ; cela ne veut dire qu'ils ne *voyaient* pas do tout : mais ils passaient là-dessus trop rapidement. Et, *pour bien voir*, il aurait fallu qu'ils prisent *la peine de le dire* » (SM : 31, our emphases).

Hegel describes that "peine" as the patience of the concept. Mathematics needs urgently that patience. To prove what is obvious, to demonstrate what is seen at a glance. What would be mathematics without such intuitive vision? And what would be mathematics without such demonstrative patient procedures?

Philosophy also needs patience. Confronted with the velocity of science, always ready to jump from one truth to another, philosophy is that infinite activity which, patiently, accepts to return, to revisit, to reexamine, to ruminate, to repeat each day the gesture of Plato's remembrance.

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To conclude, let me sum up the several concepts of intuition in Poincaré's philosophy of mathematics which were identified**.**

In the controversy with logicism, we have identified 1) a sensible intuition as the direct contact with a material entity and with its figurative representative sketched in the exterior world diagrammatic intuition (1a), 2) a (Kantian) imaginative intuition able to spontaneously produce the images necessary for thinking, 3) a conceptual intuition giving us objective knowledge of mathematical idealities, 4) an architectonical intuition of the order which, at its highest level, becomes an inventive intuition (4a), 5) mathematical induction as a (Kantian) constructive, operative intuition. An intuition which might additionally be uploaded to a Kantian intuition of the pure subject as a kind of self-awareness or "intuition of unity" (5a), which Poincaré points to in some passages.

Against formalism, Poincaré thinks intuition both as a 6) Leibnizian meaning device, that which gives meaning to words and expressions of mathematics, and as a 7) foundational device, a living force of the *esprit* (Poincaré is here almost close to Bergson) which makes impossible (and finally irrelevant) any efforts of a complete formalization.

In the context of Poincaré's very peculiar conventionalism, intuition is thought as 8) an anti-conventionalist device, an instinct enabling the mathematician to choose axioms instead of accepting pure arbitrary convention. A choice which is made on the basis of the awareness of a deeper level of mathematics accessible, not so much through a kind of Platonic intuition (9) of the true, eternal roots of mathematics, but through 9) Peircean abductive intuition, which, together with our aesthetic (architectonical) capacity of seeing the organic whole (9a), becomes the root of invention and discovery.

A very last question can now be posed set up: are all these types of intuition the manifestation of a human constructive capacity, the symptom of a mankind ability for building mathematical objects, the sign of men's power of invention or are they the expression of the life of mathematics itself? As Poincaré says:

> "Il y a une réalité plus subtile, qui fait *la vie des êtres mathématiques*, et qui est autre chose que la logique » (SM : 110, our emphases).

Surely, for Poincaré, it would be impossible to discern, to disclose, to dis-cover, to ex-pose that "life" of mathematics without the help of that human capacity which is intuition. But, what is intuition itself?

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