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Epistemology and Language in Indian Astronomy and Mathematics

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Abstract This paper is in two parts. The first presents an analysis of the epistemology underlying the practice of classical Indian mathematical astronomy, as presented in three works of Nīlakaṇṭha Somayāji (1444–1545 CE). It is argued that the underlying concepts put great value on careful observation and skill in development of algorithms and use of computation. This is reflected in the technical terminology used to describe scientific method. The keywords in this enterprise include *parīkṣā*, *anumāna*, *gaṇit*a, *yukti*, *nyāya*, *siddhānta, tarka* and *anveṣaṇa.* The concepts that underlie these terms are analysed and compared with such ideas as theory, model, computation, positivism, empiricism etc. In a short second part, it is proposed that the primacy awarded to number and computation in classical Indian science led to an artificial language that did include equations but emphasized displays that facilitated calculation, as in the Bakshali manuscript (800 CE?). It is further argued that echoes of these concepts can be recognized in current science, where computation is once again playing a greater role triggered by spectacular developments in computer technology.

Introduction

It is now widely accepted that many scientific ideas flowed from one civilization to another across the Eurasian landmass over the millennia. Classical Indian astronomy took the idea of epicycles from the Greeks; pre-modern

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Europe took the idea of the present system of numerals from India (through the Arabs); ideas on trigonometry have moved back and forth; India was in close contact with Babylonia since around 2000 BCE and acquired the sexagesimal system, still in use for measuring angles and some other quantities all over the world. However, it is fascinating to consider the inverse question of what the different cultures did *not* borrow (perhaps even *refused* to borrow) from each other, in spite of close contacts over centuries. For example, the Indians who borrowed epicycles did not take to either the assumptions that underlie Ptolemy's astronomy or the axiomatic method used by Euclid. Each culture appears to borrow selectively. Tools that support or promote the agenda of the culture are more easily adopted than the philosophy (or epistemology) underlying the invention of the tools.

Differences in approach to the same subject or to organizing the same observational data (as in astronomy) are not uncommon. For example, Ptolemy's celebrated text *Almagest* (~150 CE) devotes the first of its 13 books entirely to the question of the assumptions it makes, on the basis of which the results in the rest of the text are deduced by Euclidist methods. Āryabhaṭa (5th c. CE) adopts a wholly different approach to the subject, although he uses Greek epicycles. Instead of starting with assumptions and models, he begins with lists of numerical parameters that he needs in the 60 or so algorithms described in the book. Euclid starts his book on geometry with axioms, the Śulba-sūtras start with units of length measurement.

Such facts lead one to conclude that the approach to acquiring rational, objective knowledge can be different in different cultures. Neugebauer (1975) remarks on such differences between the Greeks and the Babylonians, for example: Babylonian methods were 'arithmetical', Greek methods start with models and the kinematics of planetary motion. A fundamental difference between the approaches of the two cultures has resonance even in modern science. Richard Feynman, perhaps the greatest theoretical physicist of the second half of the 20th century, said (Mehra 1994)

> There are two ways of doing physics: the Greek (from first principles, axioms) and the Babylonian (relating one thing to another). I am a Babylonian … I have no preconception about what nature is like or ought to be.

The key epistemological issue is about those *preconceptions*.

When such preconceptions, alternatively axioms, are successful in comparisons with observation, we may have outstanding new science. When no check with reality is possible, the axiomatist approach can lead us astray. Current controversies on string theory ('it is not even wrong', it has been said) provide a striking modern example. It is necessary to realize that such differences are fundamentally epistemological, i.e. reflect different views on what constitutes valid, reliable or acceptance-worthy knowledge.

Now epistemology in general—i.e. beyond science—has been a major preoccupation in Indian philosophical systems. It is therefore interesting to explore in what way the practice of mathematical astronomy in India was related to the *pramāṇa-śāstra*, epistemology, discussed in these systems. Most Indian astronomers do not directly expound the philosophy behind the methods they adopt, but one remarkable exception is the Kerala astronomer–mathematician Nīlakaṇṭha Somayāji. It is our purpose here to describe briefly his epistemology (a more detailed discussion will be found in Narasimha 2008a,b), and more particularly to examine the key technical terminology that he uses for this purpose.

Nīlakaṇṭha Somayāji: The Background

The life and work of Nīlakaṇṭha Somayāji (1444–1545) have been briefly discussed by Sarma (1976, 2002). Nīlakaṇṭha came towards the end of the creative period of the Kerala school of mathematics and astronomy founded by Mādhava (c. 1340–1425) of Sangamagrāma or Iriñjālakhuḍa, near Cochin (now Kochi). Although little of Mādhava's work is now extant, he is acknowledged to have made remarkable discoveries (most famously infinite series for trigonometric functions) several centuries before their (re-)discovery in Europe. He was treated universally with the greatest respect by all his intellectual descendents of the Kerala school.

Nīlakaṇṭha hailed from Trikkantiyur in South Malabar, and learnt his mathematics from Dāmodara, son of Parameśvara, well-known for his work on *dṛg.gaṇita* (seeing and computing—a phrase we shall return to later). Nīlakaṇṭha was well read in philosophy, having made a special study of Vedanta under a distinguished scholar, Ravi, and was called *Ṣaḍ-darśana-pārańgata* (master of the six philosophical systems) by Sundara-rāja, a Tamil astronomer. Nīlakaṇṭha's analysis of the epistemology of his subject must therefore be considered as that of a knowledgeable expert.

Among his many works, we shall pay special attention to *Jyoti*r-*mīmāmsa* (Sarma 1977a), *Siddhānta-darpaṇa* (Sarma 1976) and *Tantra-sańgraha* (Sarma 1977b). Other works of his, particularly the *Ārya-bhaṭīya-bhāṣya*, should also be relevant. A recent book celebrating the fifth centenary of *Tantra-sańgraha* (Sriram et al. 2002) is a valuable summary and analysis of this important work, and provides fascinating insights into Indian astronomical science.

Nīlakaṇṭha's long life of 101 years covered an age of global transition, including the rise of the Vijayanagar Empire of south India to its peak under Krishnadeva Raya (d.1529) as well as its subsequent fall, and the arrival of Vasco da Gama in Calicut (now Kozhikode) in 1498. Younger contemporaries in Europe included Columbus (1451?–1506), Copernicus (1473–1543) and Leonardo da Vinci (1452–1519), all of whom he outlived. Nīlakaṇṭha therefore represents Indian astronomy and mathematics in the early modern era.

The Pramāṇas

As background to an analysis of Nīlakaṇṭha's epistemology, we briefly consider the philosophical ideas prevalent in India in his times. These are discussed in an extensive literature in India on what is called *pramāṇa-śāstra* in Sanskrit, which may be loosely translated as epistemology. Roughly speaking, *pramāṇa* is a means of acquiring valid knowledge (Hiriyanna 1932), more precisely correct cognition (Chakrabarti 2003). The word *pramāṇa* is explained as meaning *pramā-karaṇa*: *pramā* may be thought of loosely as knowledge, so *pramāṇa* is 'knowledge-maker' or knowledge-source. However, the Indian concept of *pramā* is not identical with that of knowledge in English, and there are subtle differences; we cannot go into them here but refer the reader to Chakrabarti. There were six major Indian philosophical systems, the *ṣaḍ-darśana* of which Nīlakaṇṭha was a master. (Incidentally, the praise that Nīlakaṇṭha received from Sundararāja shows that there cannot have been rigid sectarian restrictions on the study of these systems, and that they generally flourished in parallel—no single system having been successful in displacing all the rest.) They devote considerable attention to the question of what constitutes valid knowledge and how it is to be derived, and held different views about it and accepted different *pramāṇas*. Table 1 lists the systems and the *pramāṇas* each accepts. Beginning with the earliest materialist *cārvāka* ('sweet-speech') school, we go down the table in time, and in the order in which a new *pramāṇa* was added to the earlier system.

Let us note four striking characteristics from this list.

1. All the systems accept observation, *pratyakṣa*, as the first *pramāṇa.*
2. All except the first system (whose purely materialistic view was rejected very early in the Indian history of ideas) accept *anumāna* or inference as the second *pramāṇa.*
3. None of them makes direct mention of deductionist logic in the western sense of the word as a *pramāṇa.*
4. Each system adds one more *pramāṇa* to the previous entry in the list.

The *pramāṇa* that has aroused greatest controversy in Indian scientific thinking is the third. If it is taken as *śabda*, generally interpreted as the Vedas or authentic scripture, it demands a subservience to received text that many scientists found difficult to subscribe to (we shall return to this question below). On the other hand, past data plays an important role in astronomical research, for a model or *siddhānta* cannot be built up entirely on one's own

Table 1 The six Indian philosophical systems

System	*Pramāṇas*
1. Lokāyata	*pratyakṣa* (observation)
2. Vaiśeṣika, Bauddha	+ *anumāna* (inference)
3. Sāṁkhya	+ *āpta-vacana* (trusted testimony)
4. Nyāya	+ *upamāna* (analogy)
5. Pūrva-mīmāmsa (Prabhākara)	+ *arthāpatti* (presumption/hypothesis)
6. Pūrva-mīmāmsa (Kumārila Bhaṭṭa)	+ *anupalabdhi* (non-apprehension)

Serial number = number of *pramāṇas*

personal observations. A third *pramāṇa* as in the *sāṁkhya* system, namely *āpta-vacana*, testimony from a reliable source, is therefore essential.

Anumāna is the key reasoning process in Indic thinking—philosophical or scientific. It is often illustrated by the so-called (or so mis-called) five-member syllogism. In the familiar three-member Aristotelian example of the Greek syllogism about the mortality of Socrates, the logic would survive even if mortal were replaced by immortal. This is because here the validity of the logical process is the concern, not whether the conclusions (or the premises) are sound. In the stock example of *anumāna*, the Indian logician is interested in whether there *is* fire on a particular hill. For our limited purposes here it is only necessary to note that the *observation*

There is smoke on the hill

is one essential part of the syllogistic, and the other is the *association*

Whenever there is smoke there is fire.

These support the inference or *conclusion*

There is fire on the hill.

This is of course a vastly oversimplified version of Indian thinking on inference, and various refinements, additions and deletions, intended to tackle potential pitfalls in the inferential process, have been extensively discussed (e.g. Matilal 1998; Ganeri 2001; Staal 1988).

Hiriyanna (1932) suggests that the attention devoted to *pramāṇas* in the *darśanas* probably arose from a need to counter the heterodox positions of Buddhist and Jain thought, which claimed to be based on reason. It therefore became necessary for the parties involved in the controversies to explicitly state their position on how validated knowledge was to be derived. In other words, the controversies went beyond possible flaws in logic to the fundamental principles that govern how 'knowledge' becomes worthy of acceptance. So the *pramāṇas* accepted by each system, i.e. its epistemology, had to be declared openly, and served (so to speak) as the 'axioms' accepted for rational thought within the system.

Given the pluralist approach implied by the simultaneous presence of multiple systems, it is perhaps surprising that there appears to be an overall preference in Indian thought for three *pramāṇas*: *pratyakṣa, anumāna* and some kind of testimony—*āpta-vacana* in the *sāṁkhya* system, *śabda* in the more orthodox ones. We shall return to this point later.

Nīlakaṇṭha's Epistemology

Against this background we summarize briefly here the main features of the methodological approach of Nīlakaṇṭha to his work (a more detailed discussion will appear in Narasimha 2008a,b). Many of the ideas proposed in this work are attributed to earlier authorities (often with citations—Nīlakaṇṭha was a true scholar), but for the purposes of this paper it is not always considered necessary to cite the original source.

Strategy for Establishing an Astronomical Knowledge System

Nīlakaṇṭha's objective is the development of a validated knowledge system. This emerges from a process that involves collective effort over several (intellectual) generations. The major elements of the process are the following (JM 3):

– *parīkṣā*,
scientific observation [see below]

– *gaṇit*o-*'nnītasya … pratyakṣeṇa sam̀vādaḥ*,
comparison [see below] of computational results with observation

– *tat*o *gaṇita-ling.*o*padeśaḥ,*
instruction [to disciples in the 'school'] of the [associated] computational image (algorithm?)

– *tata*s.*tasy′.āpt-*o*padeśa-avagata. anvayasya … anumānam,*
inference after instruction to trusted disciples,

– *samvādah, parasmai c′-*o*padeśaḥ.*
[more] comparison, instruction etc.

The word that has been translated as 'comparison' here is *sam̀vāda*, literally conversation or friendly debate, and that is what Nīlakaṇṭha wants between computation and observation. Finally he says,

– *iti sampradāya.avicchedāt prāmāṇyam*,
from such an unbroken tradition [is] reliable knowledge [established].

What is remarkable about this view is that it presents a multi-generational strategy for creation of validated knowledge. The process chiefly involves the establishment of a 'school' which trains new disciples, teaching them (in one or more *gurukulas*) about making careful observations, mathematics and continual comparison between the two. The emphasis is not on the sudden discovery of 'truth' by the brilliant or inspired individual (although there were several such individuals in the Kerala school), but on the development of a system by a voluntary group over a period of time. The components of the strategy are *parīkṣā* (observation), *gaṇita* (computing), *sam̀vādah* (comparison), *upadeśa* (instruction), *anumāna* (inference), *sampradāya* (tradition).

We shall discuss some of these words below, but before doing so we have to examine the question of whether there could (and would) be many schools, and they could all flourish simultaneously. The answer is clearly in favour of a hierarchical pluralism, if we go by Nīlakaṇṭha's view of the *siddhāntas*.

Multiple Siddhāntas, None Universal in Time

The scientific content of classical Indian astronomy was expressed in *siddhāntas*, which are basically validated algorithm packages (see below). Nīlakaṇṭha recognizes that there are several of these and that they do not all necessarily agree; he cites how the cry may be heard,

> *paraspara viruddhāś.ca siddhāntā bhavanti*
> the siddhāntas are in mutual disagreement with each other.

and so

> *saṅkaṭe mahati patitāḥ smaḥ*
> we have fallen into a deep crisis.

However, there can be a hierarchy among the *siddhāntas*, depending on which agrees better with observation; and the rank-ordering in the hierarchy may vary with time. Thus he cites the report of Govindasvāmi (800 to 850 CE) that Parāśara ranked them in the order *sāurya, brāhma, romaś*a, *vāsiṣṭha* and *paulīśa*, whereas Varāhamihira had ordered them differently in the sixth century, with *sāurya* still at the top of the list, followed by *paulīśa* and *romaka* during this period. Over time *paulīśa* had therefore slipped from number 2 to bottom of pile.

Going back to the crisis created by discrepancies among the different *siddhāntas*, Nīlakaṇṭha recommends that under such conditions more observations need to be taken with instruments and compared with calculation, and that the numerical parameters should be changed (or the algorithms tuned) so as to improve agreement. In other words a new *siddhānta* has to be created (JM6):

> *abhinava-siddhāntaḥ praṇeya.*

Siddhāntas are thus human creations, and the best at any time may not remain so for long—it is valid only for some finite periods of time (*kvacit kāle pramāṇam*, JM6).

Nīlakaṇṭha's approach is therefore both tentative and pluralist, but carries an objective criterion by which different *siddhāntas* may be rank-ordered in a hierarchy.

Observation and Inference

The crucial importance of *parīkṣā* and *anumāna* is highlighted by Nīlakaṇṭha. His *parīkṣā*, standing for careful observation (see section "The Keywords in Nīlakaṇṭha's Works"), can be seen as an extension of the concept of *pratyakṣa*, the first and oldest *pramāṇa* in Indian epistemology. Earlier authorities like Āryabhaṭa and Bhāskara are cited to support the importance of *parīkṣā*. Indeed, following Bhāskara I (6 CE) Nīlakaṇṭha makes the extraordinary statement (JM5)

> *avidita-sarvasya parīkṣay''.*aiva *sarva-jñānaṁ*
>
> All knowledge about everything unknown [comes] only from careful (scientific) observation.

And there is a reference to a *parīkṣā-sūtra* (JM:5).

Anumāna often appears in conjunction with *parīkṣā*, *parīkṣaṇa-anumāna* and *pratyakṣeṇa-anumānena* (JM:4). This coupling of the two is strongly reminiscent of the western idea 'By experience and by reason'.

Nīlakaṇṭha quotes both philosophical and mathematical authorities (respectively Kumārila Bhaṭṭa (~700 CE) and Bhāskara I (~600 CE)) as endorsing the fundamental importance of inference.

As the key reasoning process in Indic science, *anumāna* receives observation, carries out reasoning/calculation and whatever else is necessary to deliver validated conclusions. Thus the first two *pramāṇas* of the *darśanas* are accepted, with a slight enhancement of the first to include careful—and when necessary instrument-aided—observation.

The Centrality of Yukti

Yukti (loosely speaking 'skill', see discussion below) is a highly prized value in Indian science. Nīlakaṇṭha praises his hero Āryabhaṭa as *sarva-yukti-nidhi*, a treasure-chest of all *yukti* (JM41); Bḥāskara II praises *bīja-gaṇita* (algebra) as *avyakta-yukti*, *yukti* in handling *avyakta*, unknown variables (*Bīja-gaṇita* 2). The key role played by *yukti* is highlighted in a striking way in the commentary written by Nīlakaṇṭha on his own slim work *Siddḥānta-darpaṇa.* He first provides a demonstration of *Bhujā-koṭi-karṇa-nyāya* (the logic of the base, height and diagonal [of a rectangle]), i.e. the theorem of Pythagoras. (In India this theorem is generally expressed in terms of the diagonal in a rectangle rather than the hypotenuse of a right-angled triangle; we shall call the result the theorem of the diagonal.) Nīlakaṇṭha then makes a statement that, if taken as a more general dictum, is powerful (SD 24):

> *etat.sarvaṁ yukti-mūla*m.*eva, na t*v.*āgama-mūlam*
> All this is rooted in *yukti*, not in the *āgamas*.

Now the *āgamas* (=what has come down [from the past]) usually represent traditional, sacred texts. So Nīlakaṇṭha is rejecting the notion that 'all this' has come from some sacred text, and that it can instead be demonstrated purely by '*yukti*', skillful argument by human intelligence—as he says, without the use of hands, eyes etc. (*cakṣ*u*r.hastādi-vyāparaṁ vinā*). What does 'all this' include? Was this dictum of Nīlakaṇṭha just a passing remark at the end of a mathematical demonstration, or did he intend to make a more general point?

In the context in which the statement appears, only the result on the square of the diagonal might seem to have been intended. However, it is possible that he was using the result only as an example to illustrate how rather surprising conclusions can be drawn with clever, rational arguments, in the context of a more general controversy raging in India at various times, regarding the differences between purāṇic and siddhāntic views. Perhaps the most famous instance of this controversy is the difference of view between Āryabhaṭa and Brahmagupta about the cause of eclipses. Āryabhaṭa is clear that eclipses are a matter of shadows, whereas Brahmagpta upholds the traditional (purāṇic) *rāhu-ketu* theory, and in fact is scathingly critical of Āryabhaṭa. This conflict between the two great astronomer–mathematicians was commented upon by Al-Biruni (Sachau 1964).

The possibility that Nīlakaṇṭha was making a more general remark appears less far-fetched in the light of the work on the course of cosmological controversies in early–modern times by Minkowski (2004). There were those who held the differences to be irreconcilable, but the vast majority of astronomers sought to find explanations for the differences. Minkowski describes the attempts made to reconcile them (*virodha-parihāra*), especially with respect to such purāṇic views as that the earth was flat. These varied from the view that the *purāṇas* were for salvation and the *siddhāntas* were for worldly affairs (*vyavahāra*), so their domains were different; that the *siddhāntic* view that the earth was round was not really in conflict with the *purāṇic*, whose view must be seen as describing a locally flat picture; that the two views might have been valid in different ages, that they provided different layers of truth, that calculations were not reliably provable, and so on. Some views, like Bhāskara's connecting the earth's stability and its gravitational power, were criticized for being circular arguments. There were attempts to relate where the *siddhāntic* views stood in relation to classical *pramāṇa śāstra*, especially when the latter accepted *śabda* (Vedas) as one *pramāṇa*.

These debates ranged around Nīlakaṇṭha's times, certainly in the north. In the light of these facts, it does not seem fanciful to imagine that Nīlakaṇṭha's approval of *yukti* at the expense of *āgama* was indeed in support of the siddhāntic or rational view as against the *purāṇic* view.

Constructionist Demonstration

The demonstration of the theorem of the diagonal, offered by Nīlakaṇṭha in the *Siddhānta-darpaṇa*, is constructionist. The method used is called *kṣetra-cchedana*, literally areal dissection or division: it is a 'cut and paste' procedure (described in Narasimha 2008a,b). It is not a 'formal' proof in the sense that the word is currently used, for there is no statement of axioms, either self-evident or contrived, from which the conclusion is derived by deductionist logic. Before setting out the procedure Nīlakaṇṭha says the result is obvious to intelligent people (more precisely those with extraordinary intuition: *pratibhā-juṣām*), and provides the demonstration only for the dull-witted (*mandamati*).

The Keywords in Nīlakaṇṭha's Works

The key concepts underlying Nīlakaṇṭha's epistemology are reflected in its technical language, in particular the several powerful words that constitute its terminology. Among these words we may specially mention *parīkṣā, anumāna, gaṇita, siddhānta, yukti* and *anveṣaṇa*. We shall now consider these in turn.

Parīkṣā

As we have already seen, *pratyaksạ* (observation/perception) is the first *pramāṇa* in all six Indian systems. The word *parīkṣā* is commonly taken to mean examination or test (e.g. *agni-parīkṣā*, ordeal/test by fire). The etymology is *pari* + √*īkṣ*; *pari* being around, fully, in the direction of, towards, against, opposite; *īkṣ* = to see. So the verbal root is: to inspect something carefully from all angles or in all aspects. In the context of astronomy, *parīkṣā* can therefore stand for careful comprehensive observation as well as for testing, verification etc. Nīlakaṇṭha urges at one place (JM6) *yantraiḥ parīkṣya*, which can only mean 'having observed using instruments'; at another he talks about *samyak-parīkṣā*, 'proper' *parīkṣā*. In *Siddḥānta-darpaṇa* (SD:20) he says that one way of understanding the second part of the book is by examination or anlayses of observation, *pratyakṣa-parīkṣaṇaiḥ*; the two words are used in combination. Furthermore *parīkṣā* requires training, and imparting it is a major objective of instruction and *śāstra* (JM:8). So *parīkṣā* is not mere observation, but trained, instrumentally aided, carefully weighed and considered observation. For all these reasons, it seems appropriate to consider *parīkṣā* as standing for *scientific observation*, a sophisticated version of *pratyakṣa* In the context of astronomy, it does not seem appropriate to translate the word as 'experimentation', as Sarma (1977: xv) does; for experimentation implies a control over the conditions under which it is carried out, and this is impossible in astronomy.

Anumāna

Anumāna, inference, is the second most commonly used *pramāṇa* in Indian philosophy. It is the key reasoning element in Indian epistemology, the process that organizes the large number of 'facts' or inputs from *pratyakṣa* and *parīkṣā* into fewer, more general entities or principles. The word is derived from *anu*, = following [something else], *māna* = [a] measure, standard; so *anumāna* is a conclusion arrived at according to, following, or derived through a standard [procedure]. There is a vast literature on Indian logic and inference, and it is of course not our intention to go into it here at any length. What we need to note for our purposes here is that inference is not a formal deduction from a given set of axioms, but rather a derived conclusion from observation and, certainly in mathematical astronomy, calculation. To that extent the process involved in formal deduction is more certain (because it is fundamentally tautological, as Wittgenstein (2001) noted), but there is the uncertainty in the correspondence between the axioms and reality; the process involved in inference is less certain to the extent that the data, the 'facts', are never complete, but proceeds from more solid, certain premises. There is nothing in the process of inference that precludes the creation of multiple *siddhāntas*, or their modification over time as more 'facts' become available. This view is consistent with the analysis of Matilal (1999), who emphasizes that Indian logic is primarily a study of inference patterns, and includes certain epistemological issues, whereas modern logic excludes them.

This may be the right place to consider *tarka*, which is often offered as an Indian form of logic. It is basically argument of a particular type, characterized by Matilal (1999) variously as hypothetical, indirect, supportive. According to the commentary *Nyāya-bhāṣya* of Vatsyāyana (c. 350 to 425 CE) on the *Nyāya-sūtra* of Akṣapāda Gautama (c. 150 CE) and of Uddyotakara on Vatsyāyana, *tarka* is reasoning based upon some a priori principle. Both commentators warn that such reasoning cannot yield any empirical knowledge (the point that Wittgenstein more recently made), as it is directly dependent on a priori assumptions, preconceptions, unvalidated hypotheses. For this reason *tarka* is not considered a *pramāṇa*, although it may play a supportive role, and can be useful in detecting inconsistencies in the adversasry's arguments. Matilal gives an example of the negative use of *tarka* in the *reductio* style:

– If A were not B then A would not have been C.
– But it is absurd to conceive of A as not-C (for it is inconsistent with our standard beliefs or rational activity).
– Hence A is B.

(Perhaps the last line here should read—Hence A cannot be not-B.) It is therefore clear that *reductio* arguments were known in India, but they were used to reject the antecedent because of the absurd consequent, rather than to assert the truth of a position. It is this negative view of hypothetical reasoning that made the axiomatist approach unacceptable in Indian logic. *Tark*o-' *pratiṣthaḥ* ('tarka/logic is not well founded') is a statement which appears in many Indian texts, including the *Mahābhārata* (in the famous *Yakṣa-praśna* episode) and in Śaṁkara's commentaries on the Upanishads.

Translating *anumāna* as inference is appropriate because inference is the process by which, based on past experience (observation, in particular), we construct an objective system that generally allows us to offer information about events and objects outside our domain of experience, in the past, present or future. We have necessarily to admit that the information so offered may not necessarily be correct, e.g. a prediction may not agree with (subsequent) observation. We have then to revise the system earlier constructed, and continue the effort to devise a more effective system.

The above description of the inference process encapsulates Nīlakaṇṭha's view about how a *siddhānta* is to be devised in astronomy, and also such views about scientific inference as were expressed e.g. by Jeffreys (1931).

Gaṇita

Gaṇita, derived from *gaṇ* = to count, is literally counting or reckoning, but generally used for mathematics. (Incidentally the word mathematics is itself derived from a Greek root meaning learning, intelligence, cognate with the Sanskrit *medha*.) As the word indicates, the Indian view of mathematics was tied to number or, as Hermann Weyl (1950) put it, the concept of number was considered logically prior to the concepts of geometry in India. Even treatments of geometry, as e.g. in the *Śulva-sūtras* (8 c. BCE), have a strongly numerical flavour. Logic was a different subject altogether, only indirectly connected with mathematics, in contrast to Hellenistic views. Deductionist logic is almost never mentioned, and correspondingly formal proofs based on an arbitrary or a presumably self-evident set of axioms is absent. Instead, there is a strong tradition of *validation* and constructivist demonstration; we can say if we wish that the Indian culture of proving was different—an issue we shall return to.

A powerful compound word that combines two concepts is *dṛg.gaṇita* 'seeing and computing'. Nīlakaṇṭha talks about the objective of mathematical astronomy as *dṛg.gaṇita-sāmya*, agreement between the seen and the computed (JM:7). It is not that models are banished but they are not primary; models should not be 'preconceptions' (~axioms?), but should be *inferred*—from observation and computation. It is convenient to think of this philosophy as computational positivism (Narasimha 2003).

Siddhānta

Classical Indian astronomical science was expressed essentially through *siddhāntas*. The word *siddhānta* is often translated as 'theory', but literally it means effective/successful/validated conclusion. For example, Nīlakaṇṭha concludes his demonstration of the theorem of the diagonal by saying '*ataḥ siddham… iti*': the demonstration is complete, the result is obtained by a proper methodology of reasoning. No claim is made regarding the 'truth' of a *siddhānta*, nor does it necessarily enunciate any 'laws of nature' and proceed therefrom. In today's terminology, an astronomical *siddhānta* would be a verified, validated algorithm-set (or package). (In today's terminology verification assures us that the intended calculation is correctly carried out, validation assures us that it compares well with observation.)

Their multiplicity, and the possibility of improving them over time, is a consequence of this view of *siddhāntas* and the central role they played in astronomy.

Yukti

The word *yukti*, like yoga, is derived from the root *yuj*, to yoke (the English verb with which it has an etymological connection), unite, put together, associate etc. It has many diverse, powerful connotations, all connected in some way with the idea of putting things together to reach a conclusion or draw an inference, for example. *Yukti* also stands for skill, ingenuity; it may be a clever trick. To summarize, *yukti* has come to mean a process, device, tool or expedient, going all the way from common sense and intelligent reasoning at one end to cunning at the other. More generally it is skillful, ingenious, smart practice. When Pingree (2003) said that in deriving the infinite series for the sine of an angle Mādhava 'relied on a clever combination of geometry, algebra, and a feeling for mathematical possibilities', he was providing as well as describing an excellent example of the power of *yukti.*

Nīlakaṇṭha's dictum at the end of his demonstration of the theorem of the diagonal, already quoted, reflects the great Indic respect for *skill.* There is the famous saying of the *Bhagavad*-*gītā* (2.50)*, yogaḥ karmasu kauśalam*, yoga is skill in action. The Buddhists spoke about *upāya-kauśalya*, skill in means. The dualists said *anumā yukti*r.*ev.oktā*, connecting inference with *yukti*. Behind this admiring presentation of *yukti* is a belief or faith in the enormous, overwhelming power of ingenuity in solving a variety of practical as well as philosophical problems.

Anveṣaṇa

This word also appears in the commentary written by Nīlakaṇṭha to accompany his own work *Siddhānta-darpaṇa*. One statement that occurs in the extended exposition of *bhujā-koṭi-karṇa-nyāya*, the theorem of the diagonal, is of particular interest. Nīlakaṇṭha first demonstrates the result for a square or an isosceles right-angled triangle (where base = normal, or bhujā = koṭi), and then extends it to the more general case. For doing this, he says, there are two ways. These are discussed by Sarma (1976) in his introduction to the work in the following terms:

> In the course of this discussion, he makes a mention also of the two methods of approach for solving mathematical problems, viz., that of logical reasoning and that of demonstration on the board. He adds that one should first try the method of demonstration, for logical reasoning is limited, endless and sometimes inconclusive (*alpa-viṣayatvāt ānantyā*d. *kvaci*d.*apy.aviśrānteś.ca* …).

Such a rejection of 'logical reasoning' is generally consistent with the Indic approach to mathematics described earlier.

Ramasubramanian (2006, personal communication) translates the criticism of the 'first method' differently, interpreting it as enumerative—presumably listing all (integral?) Pythagorean triplets. The use of the word *saṁkhyā* in the passage in question may encourage this interpretation. However, the result does not apply only to integers, and the existence of Pythagorean triplets will not automatically prove the theorem of the diagonal.

One thing that however seems clearer in the same passage is that Nīlakaṇṭha considers what he calls *anveṣaṇa*—whether of the enumerative or of the logical kind—as of no substance and endless. At first sight this seems surprising, as the word is commonly thought to mean quest or search, but it was used to mean circular argument in *sāṁkhya* literature (Narasimha 2008a,b). Such a view is consistent with our earlier discussion of *anumāna* and *tarka.*

Some General Remarks

Exact equivalents for certain common western concepts like theory, axiom and (formal, deductionist) proof are hard to find in Indian languages. This is consistent with an epistemology that attached value solely to agreement between observation and computation. The absence of deep commitment to particular theories and models, and to concepts like the Greek preference for 'perfect' figures like a circle or a sphere, enabled Indian astronomers to get more accurate predictions than Ptolemy did. This is shown by the detailed comparisons made on nine astronomical parameters by the British mathematician Playfair in 1790. An epistemology that held that *siddhāntas* were basically tentative, and could be improved in time to match new observations, yielded superior accuracies till about a hundred years after Newton. Thereafter, however, with improved computational and analytical techniques, Newtonian mechanics quickly surpassed the accuracy of Indian methods.

Correspondingly, it is difficult to find exact equivalents in English for such Indian concepts as *dṛg.gaṇita*, *yukti* and *siddhānta.* The terminologies in use reflect the epistemology that informs the scientific effort in each culture. All scientific cultures seek consensible, objective systems that agree with observation, but in different ways. They adopt different proving cultures, and establish knowledge systems whose value generally varies with time and place.

Artificial Language in Indian Mathematics

Apart from the special terminology devised to describe the epistemology of Indian mathematics as we have discussed above there are other matters that also have to do with some kind of artificial language. One is concerned with representing numbers in literary texts. There is the well-known *kaṭapayādi* system, in which the numbers 1–9 and 0 are represented by a consonant (ordered as in the Sanskrit alphabet) immediately followed by a vowel, and 0 also by a free-standing vowel. A multi-digit number can therefore be a word with a literary meaning and can often be introduced into literary text. Āryabhaṭa devised a system with consonants representing numbers and vowels representing powers of 100 (see e.g. Narasimha 2001), enabling him to state a table of 24 sines, for example, in a single śloka. In this system the syllables representing the numbers have no literary meaning. There was also the *bhūta-saṁkhyā* system in which a word stands for an associated number; e.g. sky, being empty, stands for zero, earth or moon (being unique) represents one, hands or feet stand for two, Vedas for four etc. Such systems were necessary because much mathematical writing in the primary texts was in Sanskrit verse, with its strict demands on metre; flexible, literary or alphabetical representations of number were therefore invented. These systems have been discussed at length by Datta and Singh (1962).

Indian mathematics used artificial languages in at least two other ways. The first was in the form of equations that state relationships between numerical quantities in artificial symbols, both for the quantities themselves and for the operations to be carried out on them. The second was in the form of what I shall call displays, whose objective was to facilitate computation. We shall discuss these in turn.

Equations

Equations have a long history in Indian mathematics, going back at least to Brahmagupta, who calls them *samī-karaṇa* or *sama-karaṇa* in the *Brahma-sphuṭa-siddhānta* (BSS, 628 CE). (These words translate accurately into equation.) Pṛthūdakasvāmi (+860), in a commentary on the BSS, discusses how to form equations, and using symbols make the two sides equal. Datta and Singh (1935) discuss this history in Chap. III of their work. The importance of using symbols was clearly recognized, both for the clarity they bring to a statement and for their facilitation in calculation and (most interestingly) in demonstration. Here is how Bhāskara II (+12th c.) begins *Bīja-gaṇita*, his famous work on algebra:

> … *vyaktasya kṛtsnasya ta*ḍ.*eka-bīja*m.*avyaktam* … (*Bīja-gaṇita* 1)
>
> The source (*bīja*) of all calculation with knowns [= arithmetic] is that with unknowns.

He also says elsewhere:

> *bījam mati*r.*vividha-varna-sahāyinī hi mandāvabodha-vidhaye*
> *vibudhai*r-*nijādyaiḥ* । *vistāritā* … (*Bīja-gaṇita* 150)
>
> *Bīja* is [requires] intelligence, and [the effort needed by] duller intellects is assisted by the *varṇas* devised for their instruction by ancient scholars.

Here *bīja* and *bīja-gaṇita* are algebra, and *varṇa* refers to the letters of the alphabet or the words for different colours that were used as symbols for unknown quantities, operators etc. There was also the view that algebra was computation with demonstration, and that this is what distinguished it from arithmetic.

A particularly intriguing case is that of the Bakhshali manuscript (BM), discovered in 1881 near Peshawar (not too far from classical Taxila or *Takṣa-śilā*) in today's Pakistan. Although its date has been controversial—Hayashi (1995) suggests 800 CE—this is the oldest Indian mathematical manuscript extant today. The interest in the manuscript arises from its use of mathematical symbols, mostly for algebraic expressions, but some are or come close to being equations.

Table 2 gives an illustrative extract largely taken from Hayashi's exhaustive list of symbols used in the expressions in BM and elsewhere.

It will be noticed that there are some peculiarities. The usual arithmetical operations are generally denoted by the first syllable of the Sanskrit word for the operation; furthermore they *follow* the quantities operated on. The use of the first letter or the first few letters of a word for an operation is in retrospect not so unusual; modern mathematics uses many symbols of this kind, e.g. lim for limit, sup for supremum, and exp for exponentiation. The placement of the operator may have to do with what sounds idiomatic in natural Sanskrit. Its strangeness disappears if we think of A B yu not as A plus B but as A [and] B

Table 2 Some Indian algebraic symbols

Current notation	Indian notation	Explanation
A + B	A B yu	yu from yutam, = added
−A	A + (BM), Ȧ (later)	+ = less by, probably from letter for kṣa in kṣaya, or *ṛṇa*.
A × B, AB	A B, A B gu	gu = multiplied, from ganita
A ÷ B	A B bha	A, B dividing (or divisor).
√ A	A mū	m ū = root taken, from mūlam
x	•, yā, nīla, kālaka etc.	General unknown variable
A, integer	A1	Consistent with notation for fraction below
$\frac{A}{B}$, fraction	A B	Like modern notation, but without dividing line
Ax^2	yā va A	va = squared, from varga

Source: Hayashi (1995), Datta and Singh (1962)

added [to each other]. The plus sign denotes subtraction, and is thought to have come from the letter for kṣa, from kṣaya = loss, or from the first letter of *ṛṇa* (Hayashi 1995). Unknowns are denoted by dots, also as *yāvat-tāvat* (meaning 'of value as required', i.e. arbitrary or unknown often abbreviated to *yā*) and various words denoting colour (*kālaka* black, *nīla* blue etc.)

There are ambiguities in the notation. For example, different unknowns in an equation may not have different symbols. Thus, in the Bakhshali manuscript

$$\begin{array}{|ccccc|} \bullet & 5 & \text{yu} & \text{mū} & \bullet \\ & 1 & & & \\ \hline \end{array}$$

means $\sqrt{x+5} = y$

in modern notation, where y is an unknown for unknown x. From this point of view the dot seems to stand for [any number] 'unknown', rather than a specific unknown number.

Examples

There are several expressions in BM that can be interpreted as equations, although these are also clothed in computational terms. Take for example the following problem (folio 22 verso; Hayashi 1995, 210/315, meaning that the Sanskrit text is on p. 210 and the English translation on p. 315). The display here is

•	2	3	4	*dṛśya* 200
1	1	1	1	1

(1)

Here the thick dot stands for the unknown variable whose value is to be found. The word/symbol *dṛśya* (some times abbreviated to *dṛ*) is usually translated as visible (Hayashi 1995; Datta and Singh 1962, part 2, p. 12), and is to be interpreted as specification of the 'visible' result that follows the operations preceding the symbol, which involve the unknown x (dot) and are therefore 'invisible'. *Dṛśya*, to be seen, is a potential or future passive participle of the root *dṛś*, to see (also to know: recall that Indian philosophical systems are called *darśanas*, derived from the same root, meaning literally sight but representing (philosophical) *vision* or knowledge). So (1) says that its unknown part of the first four cells has to become seen as the known number in the last cell.

Against this background we can see that (1) is virtually an equation, which in modern notation would be written as

$$x + 2x + 3x + 4x = 200, \tag{2}$$

the solution (= *phala*) being $x = 10$. Note that in (1) x appears only in the first cell, and is part of the left hand side in (2), and is a common factor for the numbers in the succeeding cells 2, 3 and 4. We could interpret (1) as

$$x\,(1 + 2 + 3 + 4) = 200, \tag{3}$$

but must remember to include the term 1. This is so elsewhere also in BM, e.g. folio 23 verso, H 212/316.

The method of solution is instructive (this is adapted for (1) from the instructions more completely available for the similar problem in 23 verso). Take an arbitrary number for the unknown, e.g. 1. Then, in order multiply this number by the entries in each cell to get 1, 2, 3, 4. Add, to get 10. Divide the *dṛśya* by this number, 200/10. The result is the solution for the unknown, 20.

To those used to modern notation it will appear odd that the unknown appears in the first cell as a factor of not only one but the numbers in the following three cells. It is as if the second, third and fourth cells constitute the operator M−1 in the equation xM = 200. However, with some practice, the notation used by BM for such equations can be read unambiguously.

An alternative and in many ways more satisfactory system of writing equations was to write the two (equal) sides of the equation one below the other, with the terms involving powers of each unknown aligned vertically. For example, Brahmagupta/Pṛthūdakasvāmi would represent the equation that would today be written as

$$ax^2 + bx + c = dx^2 + e\,x + f \tag{4}$$

in the following fashion:

$$\begin{array}{lllll} y\bar{a} & va & a & y\bar{a}\ b & r\bar{u}\ c \\ y\bar{a} & va & d & y\bar{a}\ e & r\bar{u}\ f \end{array} \tag{5}$$

Here *yā* (= *yāvat tāvat*) would stand for x, *va* (= *varga*) for the square (so *yā va* is x^2), *rū* (= *rūpa*) for the additive number. If dx^2 did not appear on the right hand side of (4), the lower row in (5) would start with *yā va* 0.

Unlike BM, multiple variables are unambiguously handled in this system, with a different symbol for each variable. Apart from *yā*, such symbols could include *kā* (first syllable of *kālaka* = black), *nī* (similarly of *nīla* = blue) etc. Thus the modern

$$ax + by = c \tag{6}$$

would be written

$$\begin{array}{llllll} y\bar{a} & a & k\bar{a}\ b & r\bar{u} & 0 \\ y\bar{a} & 0 & k\bar{a}\ 0 & r\bar{u} & c. \end{array} \tag{7}$$

As the system would appear to have been sufficiently flexible to accommodate multiple equations in multiple variables, the basic ingredients for more powerful mathematical statements in the form of equations seem to be present.

However, we had to wait for 18th and 19th century European mathematics to develop a vastly more powerful artificial language in terms of equations (Staal 1995). It can be argued that this development (e.g. Euler's formulation of Newton's second law as $F = ma$) had to do with a fusion of the ideas of algorithm and algebra with those of natural laws—the former an achievement of a numerizing Indian culture, and the latter of a theorizing western culture.

Computation

But perhaps the most interesting role that language in the general sense of the word has played in a numerizing epistemology is the way that a calculation is arranged and organized, i.e. the way in which the numbers and the associated symbols are laid out, specifying where each input number appears, where the numbers involved in the calculations are presented and where the final answers emerge. In pre-computer days displays were a standard element of carrying out ordinary arithmetical operations. Indeed the use of numerals, and in particular the Indian system, gained part of its power from the way that the organization of the calculations could be displayed and exploited. (Clearly the scope for such organization is vastly greater in an algorist system that depends on the manipulation of symbols (at one time on a sand- or dust-board and later with paper and pencil), than with the earlier ‘abacist’ system that depended on shifting beads for performing such calculations.

In general, the Bakhshali Manuscript has a large number of displays, mostly arrangements of the numbers and operations involved in calculating or obtaining answers to the many problems set out in the book. The simplest example of such a display appears in rule-of-three (*trai-rāśika*) problems. To answer ‘If A yields B, what does C yield?’, the numbers are just laid out as the array

A	B	C

The answer is BC/A, obeying the rule ‘multiply middle and third term, divide by first’. The inverse rule of three (*vyasta trai-rāśika*) answers the question ‘If A units yield B sets (of A each), how many sets will C yield?’ The answer is AB/C—given by the inverse operation/rule ‘Multiply the middle term by the first, divide by the third’. In the array above the first and last terms are similar (same designation), the middle one is different.

Other examples will be found in Datta and Singh (1935: 210+).

The spirit of these arrays and displays survives in modern science in such examples as the notation for a matrix and, to take an extreme instance, Feynman’s diagrams. Such a diagram is a powerful method of picturising the terms in a series in a way that enables one to perform selective summations as well as to interpret the terms physically. Figure 1 shows one famous ‘Feynman diagram’, involving the creation of a positron travelling forwards in time, or

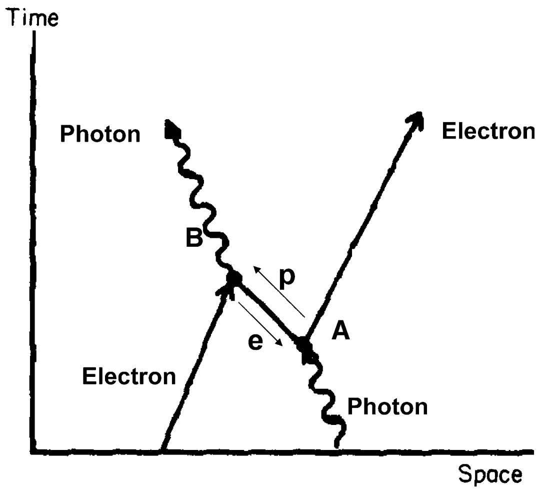

Fig. 1 A famous Feynman diagram, representing a term in a calculation that involves photon/electron/positron scattering. This is an illustration of a modern artificial language of diagrams depicting a calculation, which incidentally suggests a radical interpretation of a positron traveling forwards in time being equivalent to an electron traveling backwards in time (Feynman 1949)

equivalently of an electron travelling backwards in time. As far as the results of the calculation go there is no way of distinguishing between the two possibilities: i.e. we would not know the difference. That suggests a possibly new way of looking at many so-called scattering problems.

The use of a form of artificial language, in terms of displays and diagrams, is a subject that still needs to be studied in detail.

Conclusion

Indian mathematics did generate an artificial language of equations as well as displays or arrays intended to facilitate calculation. However, there appears to be no evidence of significant evolution in the artificial language beyond that known in the classical period. The power of the equation seen and felt in modern mathematics, where Dirac could say (as Frits Staal quotes him) that his equation was smarter than him, comes from two developments. First, beginning with Leonhard Euler (18th c.), the *laws* of mechanics formulated by Newton began to be written in symbols, in such famous examples as $F = ma$. The mathematical *structure* of this equation, as one in algebra, does not go beyond what was used in India (*kā m gu dṛ F*, where *kā* stands for (unknown) acceleration). Its physical content, however, is a totally different world: a world of theory, of universal laws, where a is given only as a second derivative of position. Again, the presence of the second derivative is in itself perhaps not so revolutionary from an Indian stand point, as the notion of first derivative was present already in Muñjāla's work (10th c.) ($\Delta\sin\theta = \cos\theta\Delta\theta$), and of second order differences in Brahmagupta's interpolation formulas. The *Suryasiddhānta* gives an algorithm for the computation of the trigonometric sines that is equivalent to solving the second order ordinary differential equation for the sine function by a finite difference scheme (Playfair 1798; Narasimha 2008a,b). It appears as if further progress was less limited by mathematical concepts and notation than by an epistemology that did not encourage a quest for universal laws in physics, which is related to the associated reluctance to look for minimal sets of axioms in deductionist mathematics. (Newton's "laws" in Latin were often translated into English as axioms; e.g. Motte 1848.)

But again, an epistemology that prized observation and calculation cannot be considered dead, if only because there are many areas where universal laws or the axiomatist equations either do not exist or are not known (e.g. economics); or, even when known, cannot be solved (e.g. in fluid dynamics, where the nonlinearity in the Navier–Stokes equations at high Reynolds numbers is so overwhelming that validated computation provides the only tool—albeit an imperfect one). With the power that modern computer technology provides, this older epistemology may become more relevant in newer, complex fields, as it has to some extent already done in work using cellular automata (e.g. Wolfram 2002).

What is an even more radical development is that computers are beginning to provide proofs where analysis has been unable to do so, as for example in the four-colour problem. It is now clear that there is more than one culture of proving, and the Euclidist method is not as privileged as it seemed a hundred years ago.

So, successful older epistemologies do not always die: they may be transformed and hybridized, but they survive, and if different and more powerful tools become available to man, the epistemology will adapt itself to exploit the new power.

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