Poincaré against the logicians: Mathematical Rigour and Culture

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Around 1906, as a response to Betrand's Russell's review of *La Science te l'Hypothese*, Henri Poincaré launched an attack on the movement to formalise the foundations of mathematics reducing it to logic. The main point is the following: the universality of logic is based on the idea that their truth is independent of any context including epistemic and cultural contexts. From the free context notion of truth and proof it follows that, given an axiomatic system, nothing new can follow. If mathematics is reducible to logic, then there is no place for creation. Philosophers would express this in the following way: logical proofs are analytic, that is provide no new information beyond the the premisses or axioms; but mathematics does provide information: mathematics is thus synthetic and hence different of logic

The dilemma seems to be linked to the notion of mathematical rigour

> 1) Mathematics is perfectly rigorous because all the inferences used in mathematical proof are of a purely logical character,
>
> 2) Mathematical proofs are not merely logical inferences. Furthermore, conclusions of mathematical proofs can, and often do, constitute extensions of the mathematical knowledge represented by the premises

One of the main strategies of Poincaré's solution to this dilemma is based on the notions of *understanding* and of *grasping the architecture* of the propositions of mathematics: mathematic rigour does not reduce to "derive blindly' without gaps from axioms; mathematical rigour is; according to Poincaré, closely linked to the ability to grasp the architecture of mathematics and contribute to an extension of the meaning embedded in structure that constitute the architecture of mathematical propositions. The focus of my talk relates precisely to the notion of architecture and the notion of understanding. I will suggest a diachronic and synchronic reconstruction of the notion of architecture - the latter considers the architecture as a cultural object. Actually what I will try to do is to link Poincaré's arguments against the logicians with his paper ***La science et les humanités*** of 1911 where he argues that the development of the ability to grasp the architecture (intuition) must be studied as the result of the refined ability of understanding acquired by means of the practice of humans sciences in a given culture.

To expresses it bluntly, according to Poincaré, mathematics is intimately related to culture because it is about the construction of a structure of relations between propositions and this structure is not universally given; but developed within the cultural conventions of a community.

1 The Problem and Poincaré's solution: Rigour in Mathematics and Rigour in Logic

In a manner reminiscent of Kant's opening remarks to the First Part of the Transcendental Problem of the Prolegomena, Poincaré opens ***La Science et l'Hypothèse*** with these words:

«La possibilité même de la science mathématique semble une contradiction insoluble. Si cette science n'est déductive qu'en apparence, d'où lui vient cette parfaite rigueur que personne ne songe à mettre en doute? Si, au contraire, toutes les propositions qu'elle énonce peuvent se tirer les unes des autres par les règles de la logique formelle, comment la mathématique ne se réduit-elle pas à une immense tautologie ? Le syllogisme ne peut rien nous apprendre d'essentiellement nouveau et, si tout devait sortir du principe d'identité, tout devrait s'y ramener. Admettra-t-on donc que les énoncés de tous ces théorèmes qui remplissent tant de volumes ne soient que des manières détournées de dire que A est A? ... Si l'on se refuse à admettre ces conséquences, il faut bien concéder que le raisonnement mathématique a par lui-même une sorte de vertu créatrice et par conséquent qu'il se distingue du syllogisme »

"La vérification diffère précisément de la véritable démonstration, parce qu'elle est purement analytique et parce quelle est purement stérile. Elle est stérile parce que la conclusion n'est que la traduction des prémisses dans un autre langage".

Poincaré [1902], p. 9-13

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2) Mathematical proofs are not merely logical inferences. Furthermore, conclusions of mathematical proofs can, and often do, constitute extensions of the mathematical knowledge represented by the premises

Important is to see that with this formulation we would like to avoid to reduce Poincaré's point to the trivial remark that the axioms of mathematics are indeed not logical but everything else follows logically from them. It looks that Poincaré links mathematical rigour with *mathematical understanding* or *mathematical insight* (perspicacité et pénétration), that is topic-specific knowledge. It is not by leaving some gaps in a demonstration that qualifies it as non rigorous but because of lack or mathematical insight (perspicacité et pénétration) or understanding of the mathematical object.

Moreover, Poincaré formulates this as a general epistemological problem. Poincaré idea is that given a set of mathematical axioms, the inferences of the mathematicians have a distinctive epistemological feature which distinguishes it from the inferences drawn by a logician from the very same axioms. Leaving by side the qualification of synthetic to the inferences drawn by the mathematician and of analytic of those drawn by the logician is that the notion of knowledge involved is different. The notion of knowledge involved by the mathematician is strongly linked with understanding the mathematical field while the notion of the logician is so to say contextually independent. In other words, knowledge of a body of mathematical propositions, plus mastery over their logical manipulation, does not amount to mathematical knowledge either of those propositions or of the propositions derived from them.

It really looks as Poincaré is aiming at a much more general epistemological point that has too close links with Kant, namely, that different kind of sciences might have a proper way of inference and because Frege's Russell's logic is based on a general notion of inference this makes it, on Poincaré's view; trivial.

But what is this mathematical understanding or insight (perspicacité et pénétration) of the mathematical object? How is this achieved? Here is Poincaré less precise and makes use of three notions that triggered important developments, namely: the notions of

- construction,
- intuition
- system or architecture.

The leading idea here is of system. Once more a Kantian topic: Each science has its own architectonic or system that consists on non logical relations between propositions. Knowledge of this architecture is knowledge to produce these relations and create new ones, here does Poincaré speak of *intuition*. A mathematical proof is related to establish a link between the architecture in which the premisses are embedded and the architecture of the conclusion. Poincaré calls this type of knowledge "intuition". Different to Kant Poincaré does not think that this architectonic is given a priori: it is a synthetic process by which the system is *constructed*. Voilà here we have the three notions mentioned above.

Certainly though challenging this is not precise enough, let me know briefly mention what Brower and the intuitionists made of these remarks.

2 The structure of the domain and the Intuitionistic interpretation

Let me express the intuitionistic interpretation and further development of Poincaré's remarks beyond perhaps his own ideas in the following way: Kant's great contribution consisted in realizing not only that mathematics as every other science has its own characteristic architectonic that systematizes it but also that mathematics has a special structured domain. On this, view, the domain of mathematics is being structured by time. Thus mathematical objects are constructions and rigorous inferences are those that always keep track of the construction of the objects the propositions involved are about. Brower interpreted Poincaré's appeal to *intuition of the structure of the domain* as *experience* of the mathematical object, meaning: the experience of constructing the object at stake.

In this framework, the proof by mathematical induction has central place: it is the most typical way of proving adequate to mathematical constructions. Proof by mathematical induction is precisely Poincaré's most cherished example of a rigorous mathematical inference that is not logical but purely mathematical.

Despite Brower's own sceptical attitude towards logics intuitionistic logicians, particularly Ardent Heyting, went a step forward and dared to describe a logical system that carries the structure of the domain to the structure of the propositions.

For the first time a logical system was not seen only as pure logical relation between propositions but as relation where the epistemic subject is introduced. Logical relations are not seen as being established by logical consequence; but by inference, where inference is the relation between propositions but between judgements, and judgements carry the epistemological structure of the domain. That is, the formal structure of inferences should be based on the constructions of the domain. In other words, mathematical objects are the result form constructions and this applies to proofs too. Time thus structured the domain of objects and the inferential relations between judgements. This has as consequences that some venerable logical axioms and logical proofs based on those axioms will fail, namley; third excluded; double negation and indirect proofs such as via absurdum.

Notice that the development of a logic that claims to be based on the idea of the structure of the domain seem to work against not only of Brower's but also against Poincaré's rejection of logics as describing mathematical proof.

The development of an inference system that carries in its structure the structure of the system of mathematics was linked too to some remarks of Poincaré where he compares the knowledge of the winning strategy in a chess game with the knowledge of the way to construct a proof. In analogous way that it is not enough to know that there is a winning strategy to win the game, it is not enough to know that there is a proof to say that mathematical proof has been performed. We must be able to show how to construct this proof. A proof beyond our abilities to construct it is not proof at all. Intuitionistic epistemologists linked this idea to the challenge of the truth as given: a truth beyond our abilities to find it can not provide the foundations of the notion of inference. It is rather the other way: human playable or reachable proof provides the foundation to the notion of inference and truth.
Michael Dummett, developed intuitionistic logic into a general conception of logic beyond mathematics and antirealism was born. Dummett and Hintikka brought into the discussion Wittgenstein's language games that provided a more precise framework to work out the notion of human playable. Indeed if language games are to work as a benchmark for the studying language and even to function as meaning mediators between language and world, these games have to be humanly playable games. Humanly playable language games were linked by the dialogical and the game theoretical tradition of Hintikka to Poincaré's and the intuitionists notion of a humanly constructible proof.

Now is that the end of Poincaré's epistemological project? Was Poincaré with the words of Brower a pre-intuitionist like Borel and Lebesgue who motivated or pre-announced the intuitionistic movement and we should go further on without him? Interesting is that Poincaré did not in fact claim against any particular logical law, but rather insisted in the notion of inference in a system and as developing the system or architecture. Let me now push this idea forward.

3 Towards a new epistemology: Structures and Modality

On my view; the very point of Poincaré's argument against the logicians is that systems of sciences are not only a set of propositions related by logical consequence. There are other; extralogical or metalogical relations which build the structure of the corresponding science. The structure of these propositions might indeed be based on the structure of the objects the propositions of that science are about, like in the intuitionistic interpretation. But the idea is more general than that and I think it could be understood when related to the recent structural approaches to modal logic

According to my reading Poincaré has a double strategy: the first strategy consists arguing from philosophy to mathematics and the second from mathematics to philosophy. From the first strategy it results that mathematics is mainly an act, a construction and from this point of view it is synthetic. From the second strategy it is an object, the result of the construction and from this point of view it is analytic and can be done in abstraction of the context where this construction was achievede. I will talk here more of the first strategy.

It is important to notice this: conditions are placed on *frames or structures*. Although models are what we deal with most often, frames play a central role. Let us recall the well known definitions of modal propositional logic

DEFINITION : Model, Frame, Truth

- A model <W,R,*v*> for modal propositional logic consists of
  1. a non empty set W of positions (traditionally interpretedc as possible worlds; contexts or scenarios: like temporal states, states of information etc.)
  2. a binary relation R on W called *accessibility relation*
  3. a valuation function *v* which assigns a truth value *v*(a) to each propositional letter of the propositional language in each position $w \in W$

- A set W with a suitable accessibility relation is called a *frame* or *structure*. Thus given a frame <W,R> we can turn it into a model by the addition of the valuation function *v*. Moreover any given frame can be turned into a variety of different models, depending on the valuation function which is added. For a frame only establishes the positions we are dealing with and fixes which are accessible from which. A valuation is needed to establish what is the case in each of the possible positions and in general there will be many ways to do that. Each of this ways is a model establishing the factual conditions under which our logical explorations will take place. The frame will provide the basis of anyone of a variety of such factual conditions.

The truth definition of modal logic tell us what formulae are true in what wi of any given model. The valuation function gives us the values of the propositional letters and the truth definition extends this to the complex formulae. The difference of this truth definition to the classical case is that the truth is here made relative to the value of the positions in the structure of the model at stake. Furthermore the evaluation is dependent too on the interrelations between the given positions in that structure.

And here we are, modal logic displays the interrelation between inference and structure. in such a way that each structure yields it's own notion of inference. Moreover, one can at the object language level display axioms that describe the structure, usually given at the metalngage level. This is called frame validity. In this framework we would say that Poincaré is searching for those inferences the result of which describe the structure. The modern modal logician would say; that Poincaré is searching for object language laws to characterize frame validity.

We should then distinguish between the logic of the model; that is, driving the logical consequences within the structure, what Poincaré might want to call the purely logical manipulation of the propositions **in** the structure (that amounts to truth in a model), and the use of propositions at the language level to describe the structure in which this propositions are embedded (truth in the frame).

Here we are assuming that the structure is given, now; let us drop that assumption. The point is then is the following. Let us assume that because of topic-related knowledge, including perhaps some no complete knowledge of the structure involved, we take that a given proposition is true and even valid, but we do not have a complete description of the structure. Then we could ask the following question: how should this structure be completed or how should it be if the given proposition has to be valid. This will takes us to a kind of inverse logic or structural abduction, that were developed recently in Lille and called: structural seeking games. These kind of inverse logic or structural abduction; seems to be related to the notion of inverse mathematics of Harvey Friedman: what would be for example the minimal higher order conditions to define the continuum as a series of Cauchy?

analitico sintético

In this field there are many others results appearing nowadays at a breathtaking pace. Let me mention one linked to the notion of human playable. One of the main motivations of researches on modal propositional logic is that it is able to manipulate in structures with a dedicidable logic important fragments of first and second order logic that are not decidable!! These parts called the guarded fragment of logic can be described and studied in decidable logical framework. One is tempted to say that those guarded fragments are the ones that are humanly playable.

But what about the domain? What happens if we would like to describe the structure of the domain. At this point we meet the famous Barcan formulae that in the philosophical tradition regulate the passage from possibility to existence but from the pureluy structural point of view descirbe one particular structure of the domain.

If the propostional frame is extended with a structure where the domains of each position at the structure are (at least) decreasing then the passage from possibility to existence is assured:

$$\diamond(\exists x)Px \rightarrow (\exists x)\diamond Px$$

If the domains are not decreasing (and not constant) then the formula does not hold. Moreover if the domains are at least increasing the inverse Barcan formula holds. That is the inverse Barcan formula describes at the object language level a so to say constructive property of the domain.

$$(\exists x)\diamond Px \rightarrow \diamond(\exists x)Px$$

Certainly if both hold then there is no construction: the domain is constant!

Now, does not the latter hold too for intuitionist first order logic? The point is here that the Barcan formulae describe the structure of the domain independently of the structure of the propositions! We have, as a way to describe the domain without touching the validity of any logical law, in the sense that no special property of the accessibility relation is assumed. A delicate case is the case of that version of the third excluded in the framework of an increasing domain. That is, if the domain is increasing, as seemed to be assumed by the constructive notion of the structure of the domain of the intuitionists and Poincaré, the validity status of the following follows:

$(\exists x)\Diamond(Px \vee \neg Px) \rightarrow \Diamond(\exists x)\ (Px \vee \neg Px)$

Moreover,

$\Diamond(\exists x)\ (Px \vee \neg Px) \rightarrow (\exists x)\Diamond(Px \vee \neg Px)$

holds too in an increasing domain

The point here is that quantifiers and non-modal operators are 'local'. Up to know; Poincaré's notions of intuition, structure and human playable seem to be somehow related to the intuitionistic interpretation. This is even the way the history took. Furthermore the notion of human playable has been more and more deepened but attaching to bounded rationality and computability. Oddly enough the creative élan; the humanity part of the project seems nowadays more and linked to development of ways of fully automating proof search. Nowadays conception of the notion of human playable seems to be understood as perfomable by machines!. One gets the feeling that this is not what neither Poincaré nor Brower were seeking for when they talked about intuition and creativity.
Unfortunatley the remarks of Poincaré and Brower are far from clear in this respect and the connections with the general epistemological project rather loose. I will thus in the next paragraph suggest some other possible sense in which intuition and human playable could be linked and will leave for further research or other more perspicuous philosophers the task of putting all the pieces of the epistemological puzzle together.

4 Verstehen und Erklären and the only human playable

Let us assume that we would leave mathematics as it is and develop some kind of method of drawing inferences. Whatever the method is this has to be formulated in the framework of higher order logic. Now, this method might able to mechanize some routine steps but do not substitute for mathematical knowledge, Poincaré would talk of "intuition". The choice of which predicat abstract to use during an application of universal rule really contains, in a compact form, the very essence of a standard mathematical argument. The point is that this indivdual choices cannot be described by a general algorithm, where this choice could be regulated as an instance of general logical case. Indeed the method of search for a given solution cannot be generalized to an infinite number of cases. In other words the choice can not be understood as the subsumption to a general law. We might be tempted to say in such framework we are not in face of an explanation but rather of understanding (verstehen): the solution has to be found for the particular case at stake. Creativity might be understood as the way to find such a solution though a general algorithm to find the solution can not be in general be provided

Astonishingly, though machines can not find a solution; human can and quite frequently find such kind of solutions. Let me take two examples: one of logic and the other of mathematics. For the former example I formalize with the help of second order logic; the proof that actually students, with some knowledge of modal logic, find. The second example embodies the principle behind many diagonal arguments in mathematics.

Let me use my favourite proof-system, namely dialogues. Actually I will not use the standard dialogues but a version that for the cases under discussion can be easily read with other proof systems in mind. Take **O**-formulae as **T** and **P** as false for example or **O**-formulae as left fromulae of a sequent calculus and **P**-formulae as the formulae at the right of a sequent. The dialogical framework might help to see the point of the "only human playable". Indeed,it is the proponents choices; who can not be put into an algorithm and requires a creative move:

Example 1: Let us think of the atomic formula Px as telling us that **P** is true at positions x, and $R(x,y)$ as saying y is a position accessible from x. Then

$$\forall yR(x,y) \rightarrow Py$$

corresponds to P being true at every position in the structure accessible from x, and hence to $\Box\boldsymbol{P}$ being true at position x. Thus $\Box\boldsymbol{P}\rightarrow\boldsymbol{P}$ is true at the position x corresponds to $(\forall yR(x,y) \rightarrow Py)\rightarrow Px$

P $\{\forall P(\forall x(\forall yR(x,y) \rightarrow Py)\rightarrow Px)\} \rightarrow \forall xR(x,x)$ 0
O $\forall P\,(\forall x(\forall yR(x,y) \rightarrow Py)\rightarrow Px)$ 1
P $\forall xR(x,x)$ 2
O $_{ki}$? 3
P R(ki,ki) 12
P? $\lambda zR(ki,z)$ 4

Here we meet the crucial choice: Proponent might choose between an infinite number of predicate abstracts and there is no general algorithm to tell us which to take. Now, with some knowledge the Proponent will choose a two places predicate abstract with a lambda bounded variable dependents on the *y* the value of which will be chosen by the Opponent.

O $\forall x(\forall yR(x,y) \rightarrow[\lambda z.R(ki,z)]y)\rightarrow[\lambda z.R(ki,z)]x$ 5
P $_{ki}$? 6
O $(\forall yR(ki,y) \rightarrow[\lambda z.R(ki,z)]y)\rightarrow[\lambda z.R(ki,z)]ki$ 7
P $(\forall yR(k,y) \rightarrow[\lambda z.R(k,z)]y)$ 8
now to subdialogues are opened

If the Opponent responds
O $[\lambda z.R(ki,z)]ki$ 9
The Proponent will ask to instantiate λz with *ki and win with move 12:*

P $_{\lambda z\,/ki}$? 10

Recall: Actually because of the particle rule of lamda at the first order level the player has no real option when he challenges a lamda operator. He has to attack with the constant the name of the predicate applies to

O $R(ki,ki)$ 13

If the Opponent responds with

O $_{kj}$? 11*

the Proponent wins with the following sequence of moves:

P $R(ki,kj) \rightarrow [\lambda z.R(ki,z)]kj$ 12*

O $R(ki,kj)$ 13*

P $[\lambda z.R(ki,z)]kj$ 14*

O $_{\lambda z\,/kj}$? 15*

P $R(ki,kj)$ 16*

Example 2:

Suppose there is a way of matching subsets of some set D with members of D. Let us call a member of D associated with a particular subset a code for that subset. It is required that every member of D must be a code and nothing can be a code for more than one subset, though it is allowed that some subsets can have more than one code. Then, some subset of D must lack a code (Cantor's theorem follows from this).

Let us formulate this in the following way, let $R(x,y)$ represent the relation: y is in the subset that has x as its code; so $\lambda y.R(x,y)$ represents the set coded by x. Then the following second order-sentence does the job

$(\forall R)(\exists X)(\forall x)\,\neg[R(x,y)= X]\}$

Or in more logical terms: that is, without identity:

$(\forall R)(\exists X)\,(\forall x)(\exists y)\{[R(x,y)\wedge \neg X(y)] \vee [\neg R(x,y)\wedge X(y)]\}$

Here the start of the proof:

P $(\forall R)(\exists X)\,(\forall x)(\exists y)\{[R(x,y)\wedge \neg X(y)] \vee [\neg R(x,y)\wedge X(y)]\}$

O $\forall/_{P}$?

P $(\exists X)\ (\forall x)(\exists y)\{[P(x,y)\wedge \neg X(y)] \vee [\neg P(x,y)\wedge X(y)]\}$

O $\exists$?

P $\neg(\forall x)(\exists y)\{[P(x,y)\wedge\neg\prec\lambda x.\neg P(x,x)\succ(y)] \vee$
$[\neg P(x,y)\wedge \prec\lambda x.\neg P(x,x)\succ(y)]\}$

…

The key feature in the proof is the use of $\lambda x.\neg P(x,x)$. This, in fact is the heart of diagonal arguments and amounts to looking at the collection of things that do not belong to the set they code. The choice of this abstract is at the centre of the required mathematical knowlegde. Everything else is mechanical. It is the nee for such choices that stands in the way of fully automating higher-order proof search.

On my view the whole movement triggered by Poincaré and Brower relates to one deep epistemological point: the core result of the building of mathematics and logic were achieved by means of the creative effort of human imagination. Mathematics and logic are creation in the same sense that art is. The challenge to fully understand the epistemological implications of this point are still there and they do not seem to stop to fascinate and puzzle us again and again.